Lemke's Algorithm:
The Hammer in Your Math Toolbox?

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First, a Word About Hammers

"If the only tool you have is a hammer, you tend
to see every problem as a nail."

Abraham Maslow

- requirements for this to be a good idea
  - a way of transforming problems into nails (MLCPs)
  - a hammer (Lemke's algorithm)
- lots of advanced info + one hour = something has to give
  - majority of lecture is motivating you to care about the hammer by showing you how useful nails can be
  - make you hunger for more info post-lecture
  - very little on how the hammer works in this hour

Hammers (cont.)

- by definition, not the optimal way to solve problems, BUT
  - computers are very fast these days
  - often don’t care about optimality
    - prepro, prototypes, tools, not a profile hotspot, etc.
  - can always move to optimal solution <u>after</u> you verify it’s a problem you actually want to solve

What are “advanced game math problems”?

- problems that are ammenable to mathematical modeling
    - state the problem clearly
    - state the desired solution clearly
    - describe the problem with equations so a proposed solution’s quality is measurable
    - figure out how to solve the equations
- why not hack it?
    - I believe better modeling is the future of game technology development (consistency, not reality)

Prerequisites

- linear algebra
  - vector, matrix symbol manipulation at least
- calculus concepts
  - what derivatives mean
- comfortable with math notation and concepts

Overview of Lecture

- random assortment of example problems breifly mentioned
- 5 specific example problems in some depth
    - including one that I ran into recently and how I solved it
- generalize the example models
- transform them all to MLCPs
- solve MLCPs with Lemke's algorithm

A Look Forward

- linear equations
  $Ax = b$
- linear inequalities
  $Ax >= b$
- linear programming
  $\min c^T x$
  s.t. $Ax >= b$, etc.
- quadratic programming
  $\min \frac{1}{2} x^T Q x + c^T x$
  s.t. $Ax >= b$
  $Dx = e$
- linear complimentarity problem
  $a = Af + b$
  $a >= 0, f >= 0$
  $a_i f_i = 0$

Applications to Games

graphics, physics, ai, even ui

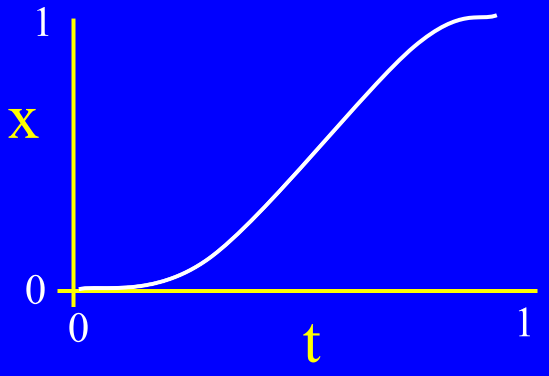
- computational geometry
- visibility
- contact
- curve fitting
- constraints
- integration
- graph theory
- network flow
- economics
- site allocation
- game theory
- IK
- machine learning
- image processing

Applications to Games (cont.)

- don’t forget...
  - The Elastohydrodynamic Lubrication Problem
  - Solving Optimal Ownership Structures
    - “The two parties establish a relationship in which they exchange feed ingredients, q, and manure, m.”

Specific Examples #1a: Ease Cubic Fitting

- warm up with an ease curve cubic
  $x(t)=at^3+bt^2+ct+d$
  $x'(t)=3at^2+2bt+c$
- 4 unknowns a,b,c,d (DOFs) we get to set, we choose:
  $x(0) = 0,\ x(1) = 1$
  $x'(0) = 0,\ x'(1) = 0$



Specific Examples #1a: Ease Cubic Fitting (cont.)

- $x(t)=at^3+bt^2+ct+d$, $\quad x'(t)=3at^2+2bt+c$

- $x(0) = a0^3+b0^2+c0+d \quad = d = 0$
- $x(1) = a1^3+b1^2+c1+d \quad = a+b+c+d = 1$
- $x'(0) = 3a0^2+2b0+c \quad = c = 0$
- $x'(1) = 3a1^2+2b1+c \quad = 3a + 2b + c = 0$

Specific Examples #1a: Ease Cubic Fitting (cont.)

- $d = 0$, $a+b+c+d = 1$, $c = 0$, $3a + 2b + c = 0$
- $a+b=1$, $3a+2b=0$
- $a=1-b \Rightarrow 3(1-b)+2b = 3-3b+2b = 3-b = 0$
- $b=3$, $a=-2$
- $x(t) = 3t^2 - 2t^3$

Specific Examples #1a: Ease Cubic Fitting (cont.)

- or,
- $x(0) = d = 0$
- $x(1) = a + b + c + d = 1$
- $x'(0) = c = 0$
- $x'(1) = 3a + 2b + c = 0$

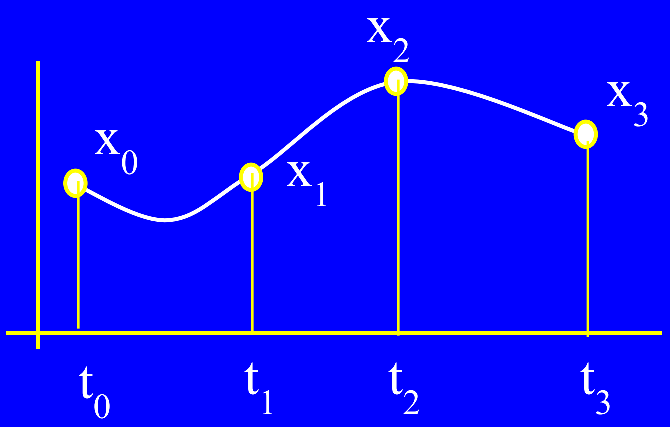
$$\begin{vmatrix} x(0) \\ x(1) \\ x'(0) \\ x'(1) \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{vmatrix} \begin{vmatrix} a \\ b \\ c \\ d \end{vmatrix} = \begin{vmatrix} 0 \\ 1 \\ 0 \\ 0 \end{vmatrix}$$

(can solve for any rhs)

$Ax = b$, a system of linear equations

Specific Examples #1b: Cubic Spline Fitting

- same technique to fit higher order polynomials, but they "wiggle"
- piecewise cubic is better "natural cubic spline"
- $x_i(t_i)=x_i \qquad x_i(t_{i+1})=x_{i+1}$
  $x'_i(t_i) - x'_{i-1}(t_i) = 0$
  $x''_i(t_i) - x''_{i-1}(t_i) = 0$
- there is coupling between the splines, must solve simultaneously



- 4 DOF per spline
  - 2 endpoint eqns per spline
  - 4 derivative eqns for inside points
  - 2 missing eqns = endpoint slopes

Specific Examples #1b: Cubic Spline Fitting (cont.)

$x_i(t_i)=x_i \qquad x_i(t_{i+1})=x_{i+1}$

$x'_i(t_i) - x'_{i-1}(t_i) = 0$

$x''_i(t_i) - x''_{i-1}(t_i) = 0$

$$
\begin{vmatrix}
\bullet & \bullet & \bullet & \bullet & & & & & & & & \\
\bullet & \bullet & \bullet & \bullet & & & & & & & & \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & & & & \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & & & & \\
 & & & & \bullet & \bullet & \bullet & \bullet & & & & \\
 & & & & \bullet & \bullet & \bullet & \bullet & & & & \\
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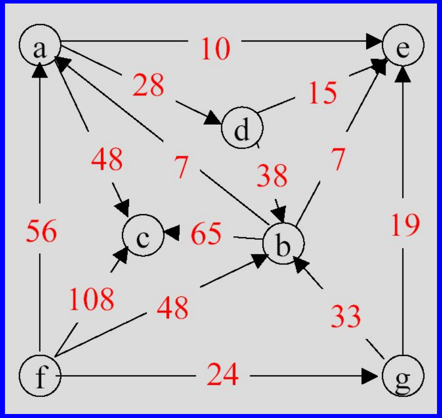
 & & & & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
 & & & & & & & & \bullet & \bullet & \bullet & \bullet \\
 & & & & & & & & \bullet & \bullet & \bullet & \bullet \\
 & & & & & & & & & & & \vdots
\end{vmatrix}
\begin{vmatrix} a_0 \\ b_0 \\ c_0 \\ d_0 \\ a_1 \\ b_1 \\ c_1 \\ d_1 \\ \vdots \end{vmatrix}
=
\begin{vmatrix} x_0 \\ x_1 \\ 0 \\ 0 \\ x_1 \\ x_2 \\ 0 \\ 0 \\ \vdots \end{vmatrix}
$$

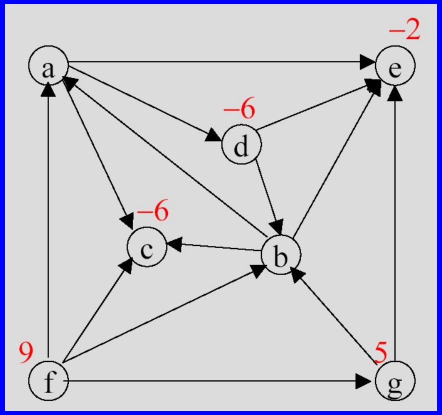
Ax = b, a system of linear equations

Specific Examples #2: Minimum Cost Network Flow

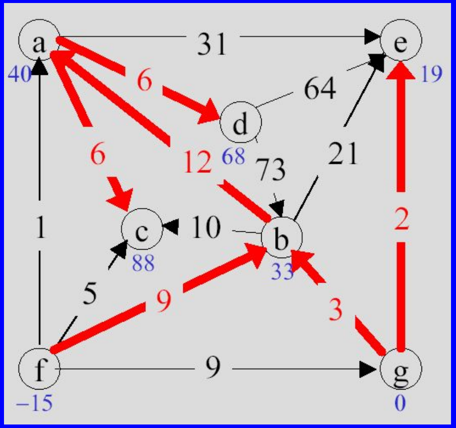
- what is the cheapest flow route(s) from sources to sinks?
- model, want to minimize cost

  c_{ij} = cost of i to j arc
  b_i = i's supply/demand, $\text{sum}(b_i)=0$
  x_{ij} = quantity shipped on i to j arc
  $x_{*k} = \text{sum}(x_{ik})$ = flow into k
  $x_{k*} = \text{sum}(x_{ki})$ = flow out of k
- flow balance: $x_{*k} - x_{k*} = -b_k$
- one-way streets: $x_{ij} >= 0$





Specific Examples #2: Minimum Cost Network Flow (cont.)



- min cost: minimize $c^T x$
  - the sum of the costs times the quantities shipped ($c^T x = c \cdot x$)
- flow balance is coupling: matrix

$x_{*k} - x_{k*} = -b_k$

$$
\begin{vmatrix}
-1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \ldots \\
0 & 0 & 0 & -1 & -1 & -1 & 1 & \ldots & & \\
\ldots & & & & & & & & &
\end{vmatrix}
\begin{vmatrix}
x_{ac} \\ x_{ad} \\ x_{ae} \\ x_{ba} \\ x_{bc} \\ x_{be} \\ x_{db} \\ \vdots
\end{vmatrix}
= -
\begin{vmatrix}
b_a \\ b_b \\ b_c \\ b_d \\ \vdots
\end{vmatrix}
$$

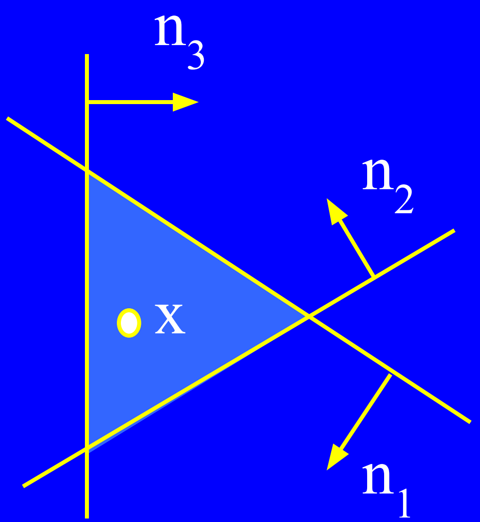
minimize $c^T x$
subject to
$Ax = -b$
$x >= 0$

a linear programming problem

Specific Examples #3: Points in Polys

- point in convex poly defined by planes
  $n_1 \cdot x >= d_1$
  $n_2 \cdot x >= d_2$
  $n_3 \cdot x >= d_3$

  $Ax >= b$,
  linear inequality



- farthest point in a direction in poly, c:

  min $-c^T x$
  s.t. $Ax >= b$
  linear programming

Specific Examples #3: Points in Polys (cont.)

- closest point in two polys

  $\min\ (x_2 - x_1)^2$

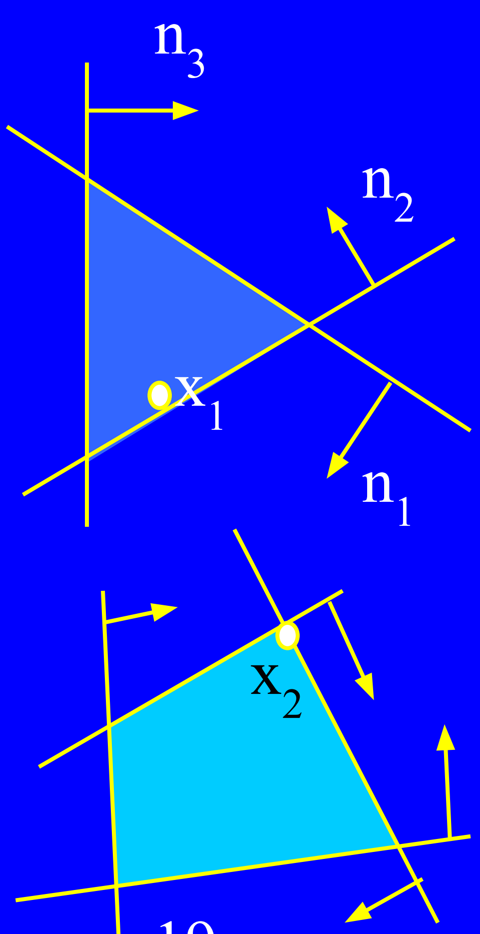
  s.t. $A_1 x_1 >= b_1$

  $A_2 x_2 >= b_2$

- stack 'em in blocks, $Ax >= b$

$$x = \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} \qquad b = \begin{vmatrix} b_1 \\ b_2 \end{vmatrix} \qquad A = \begin{vmatrix} A_1 & A_2 \end{vmatrix}$$

what about $(x_2-x_1)^2$, how do we stack it?



Specific Examples #3: Points in Polys (cont.)

- how do we stack x_1, x_2 into single x given $(x_2-x_1)^2 = x_2^{\,2} - 2x_2 \bullet x_1 + x_1^{\,2}$

$$\left| x_1^T \; x_2^T \right| \left| \begin{matrix} 1 & -1 \\ -1 & 1 \end{matrix} \right| \left| \begin{matrix} x_1 \\ x_2 \end{matrix} \right| = x_2^{\,2} - 2x_2 \cdot x_1 + x_1^{\,2} = x^T Q x$$

$x^2 = x^T x = x \cdot x$
1 = identity matrix

min $x^T Q x$
s.t. $Ax >= b$

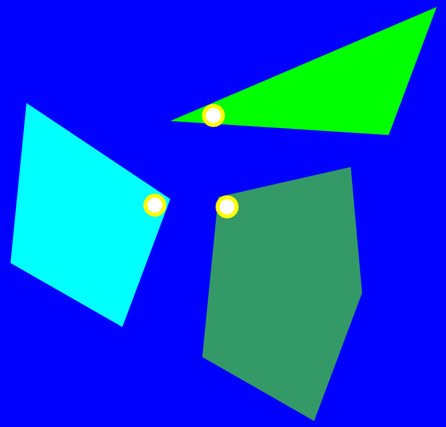
a quadratic programming problem

Specific Examples #3: Points in Polys (cont.)

- more points, more polys!

  $\min\ (x_2-x_1)^2 + (x_3-x_2)^2 + (x_3-x_1)^2$

$$\begin{vmatrix} x_1^T & x_2^T & x_3^T \end{vmatrix} \begin{vmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = x^T Q x$$



$\min\ x^T Q x$

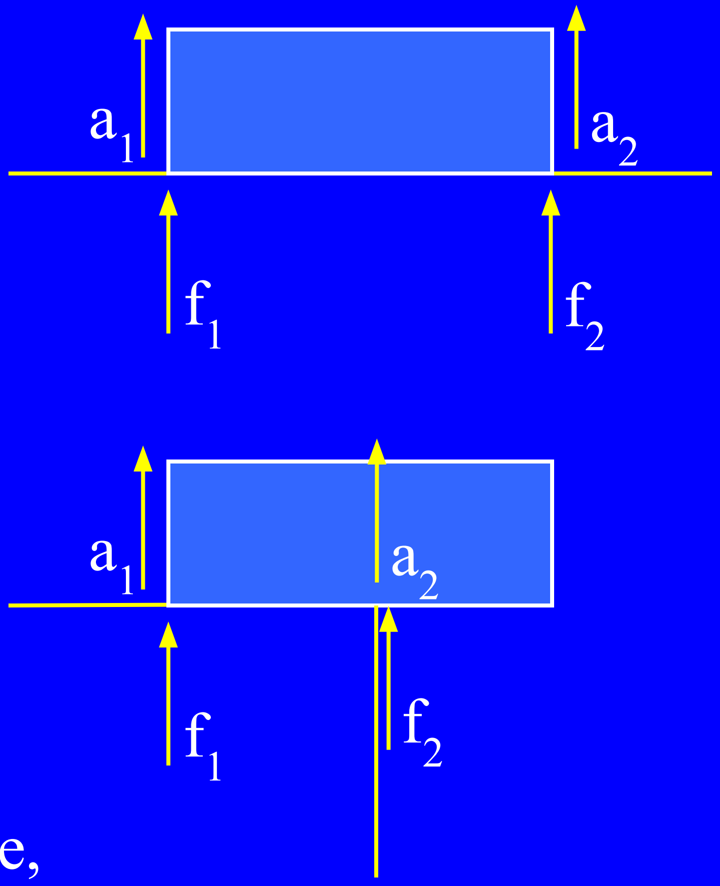
s.t. $Ax >= b$

another quadratic programming problem

- same form for all these poly problems
- never specified 2d, 3d, 4d, nd!

Specific Examples #4: Contact

- model like IK constraints
  $a = Af + b$
  $a >= 0$, no penetrating
  $f >= 0$, no pulling
  $a_i f_i = 0$, complementarity
  (can't push if leaving)

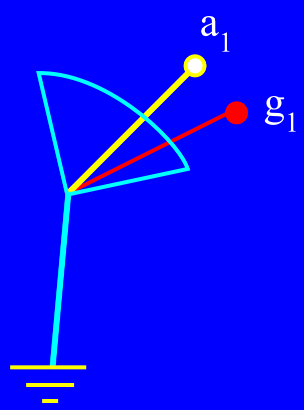


linear complementarity problem

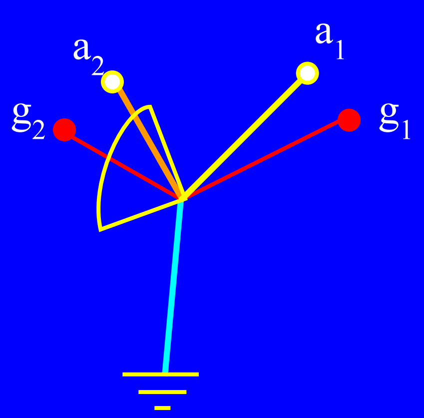
it's a mixed LCP if some $a_i = 0$, f_i free,
like for equality constraints

Specific Examples #5: Joint Limits in CCD IK

- how to do child-child constraints in CCD?
  - parent-child are easy, but need a way to couple two children to limit them relative to each other



- how to model this & handle all the cases?
- define $d_n = g_n - a_n$
- min $(x_1 - d_1)^2 + (x_2 - d_2)^2$
- s.t. $c_{1min} <= a_1+x_1 - a_2-x_2 <= c_{1max}$
- parent-child are easy in this framework: $c_{2min} <= a_1+x_1 <= c_{2max}$
- another quadratic program:
  min x^TQx
  s.t. $Ax >= b$



What Unifies These Examples?

- linear equations
  $Ax = b$
- linear inequalities
  $Ax >= b$
- linear programming
  $\min c^T x$
  s.t. $Ax >= b$, etc.
- quadratic programming
  $\min \frac{1}{2} x^T Q x + c^T x$
  s.t. $Ax >= b$
  $Dx = e$
- linear complimentarity problem
  $a = Af + b$
  $a >= 0, f >= 0$
  $a_i f_i = 0$

QP is a Superset of Most

- quadratic programming
  $\min \frac{1}{2}x^TQx + c^Tx$
  s.t. $Ax >= b$
  $\quad\quad Dx = e$

but MLCP is a superset of convex QP!

- linear equations
  - $Ax = b$
  - $Q, c, A, b = 0$
- linear inequalities
  - $Ax >= b$
  - $Q, c, D, e = 0$
- linear programming
  - $\min c^Tx$
    s.t. $Ax >= b$, etc.
  - Q, etc. $= 0$

Karush-Kuhn-Tucker Optimality Conditions get us to MLCP

$$\min \tfrac{1}{2} x^T Q x + c^T x$$
$$\text{s.t. } Ax - b >= 0$$
$$Dx - e \ \ = 0$$

- for QP
- form "Lagrangian"

  $L(x,u,v) = \tfrac{1}{2} x^T Q x + c^T x - u^T(Ax - b) - v^T(Dx - e)$
- for optimality (if convex):

  $\partial L / \partial x = 0$

  $Ax - b >= 0$

  $Dx - e \ \ = 0$

  $u >= 0 \quad u_i(Ax\text{-}b)_i = 0$
  - this is related to basic calculus min/max f'(x) = 0 solve

Karush-Kuhn-Tucker Optimality Conditions (cont.)

- $L(x,u,v) = \frac{1}{2} x^T Q x + c^T x - u^T(Ax - b) - v^T(Dx - e)$

- $y = \partial L / \partial x = Qx + c - A^T u - D^T v = 0$, x free
- $w = Ax - b >= 0$, $u >= 0$, $w_i u_i = 0$
- $s = Dx - e = 0$, v free

$$\begin{vmatrix} y \\ s \\ w \end{vmatrix} = \begin{vmatrix} Q & -D^T & -A^T \\ D & 0 & 0 \\ A & 0 & 0 \end{vmatrix} \begin{vmatrix} x \\ v \\ u \end{vmatrix} + \begin{vmatrix} c \\ -e \\ -b \end{vmatrix}$$

$y, s = 0$

x, v free

$w, u >= 0$

$w_i u_i = 0$

This is an MLCP

$$\begin{vmatrix} y \\ s \\ w \end{vmatrix} = \begin{vmatrix} Q & -D^T & -A^T \\ D & 0 & 0 \\ A & 0 & 0 \end{vmatrix} \begin{vmatrix} x \\ v \\ u \end{vmatrix} + \begin{vmatrix} c \\ -e \\ -b \end{vmatrix}$$

$y, s = 0$
x, v free
$w, u >= 0$
$w_i u_i = 0$

$$\begin{array}{ccccccc} \downarrow & & \downarrow & & \downarrow & & \downarrow \\ a & = & A & & f & + & b \end{array}$$

$a_i f_i = 0$

some a >= 0, some = 0
some f >= 0, some free
(but they correspond so complementarity holds)

Modeling Summary

- a lot of interesting problems can be formulated as MLCPs
  - model the problem mathematically
  - transform it to an MLCP
    - on paper or in code with wrappers
  - but what about solving MLCPs?

Solving MLCPs

(where I hope I made you hungry enough for homework)

- Lemke's Algorithm is only about 2x as complicated as Gaussian Elimination
- Lemke will solve LCPs, which some of these problems transform into
- then, doing an "advanced start" to handle the free variables gives you an MLCP solver, which is just a bit more code over plain Lemke's Algorithm

Playing Around With MLCPs

- PATH, a MCP solver (superset of MLCP!)
  - really stoked professional solver
  - free version for "small" problems
  - matlab or C
- OMatrix (Matlab clone) free trial (omatrix.com)
  - only LCPs, but Lemke source is in trial
    - » not a great version, but it's really small (two pages of code) and quite useful for learning, with debug output
    - » good place to test out "advanced starts"
- my Lemke's + advanced start code
  - not great, but I'm happy to share it
  - it's in Objective Caml :)

References for Lemke, etc.

- free pdf book by Katta Murty on LCPs, etc.
- free pdf book by Vanderbei on LPs
- The LCP, Cottle, Pang, Stone
- Practical Optimization, Fletcher
- web has tons of material, papers, complete books, etc.
- email to authors
    - relatively new math means authors are still alive, bonus!

Specific Examples #5: Constraints for IK

- compute “forces” to keep bones together

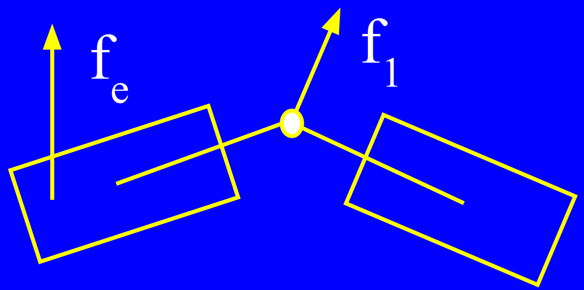
  $a_1 = A_{11}\ f_1 + b_1$

  a_1 : relative acceleration at constraint

  f_1 : force at constraint

  b_1 : external forces converted to accelerations at constraints

  A_{11} : force/acceleration relation matrix



Specific Examples #5: Constraints for IK (cont.)

- multiple bodies gives coupling...

$$\begin{vmatrix} a_1 \\ a_2 \end{vmatrix} = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} \begin{vmatrix} f_1 \\ f_2 \end{vmatrix} + \begin{vmatrix} b_1 \\ b_2 \end{vmatrix}$$

$a = Af + b$

$a = 0$ for rigid constraints

$Af = -b$, linear equations