Made by ALEX

“SCALE DRAWINGS, BEARINGS AND TRIGONOMETRY”

15 Scale drawings, bearings and trigonometry

Key words

- Scale drawing
- Bearing
- Hypotenuse
- Adjacent
- Opposite
- Tangent ratio
- Inverse function
- Sine ratio
- Cosine ratio
- Sine rule
- Cosine rule
- Projection

A scale drawing is *similar* to the real object, so the sides are in proportion and corresponding angles are equal.

Sometimes you have to draw a diagram to represent something that is much bigger than you can fit on the paper or so small that it would be very difficult to make out any detail. Examples include a plan of a building, a map of a country or the design of a microchip. These accurate diagrams are called a scale drawings.

The lines in the scale drawing are all the same fraction of the lines they represent in reality. This fraction is called the *scale* of the drawing.

The scale of a diagram, or a map, may be given as a fraction or a ratio such as $\frac{1}{50000}$ or 1 : 50 000.

Worked example 1

A rectangular field is 100 m long and 45 m wide. A scale drawing of the field is made with a scale of 1 cm to 10 m. What are the length and width of the field in the drawing?

10 m is represented by 1 cm

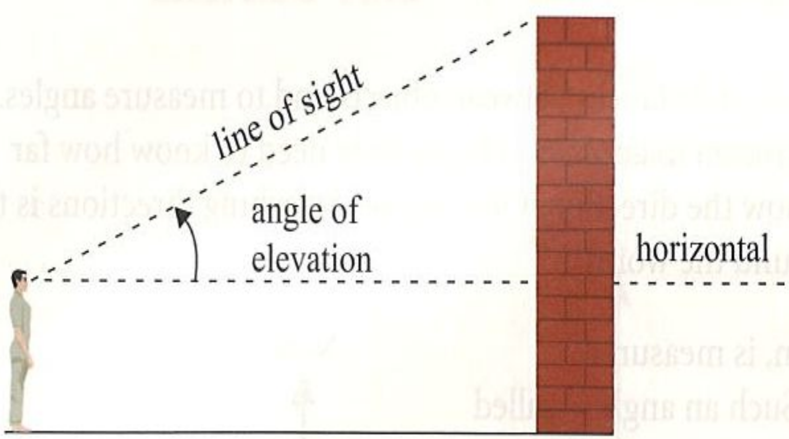
$\therefore$ 100 m is represented by $(100 \div 10)$ cm = 10 cm
and 45 m is represented by $(45 \div 10)$ cm = 4.5 cm

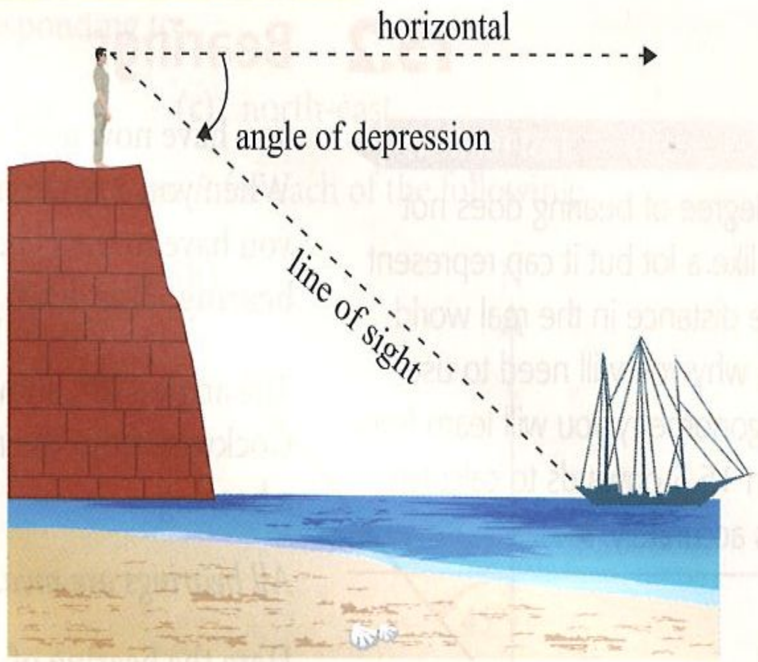
So, the dimensions on the drawing are: length = 10 cm and width = 4.5 cm.

Angle of elevation and angle of depression

Scale drawing questions often involve the observation of objects that are higher than you or lower than you, for example, the top of a building, an aeroplane or a ship in a harbour. In these cases, the angle of elevation or depression is the angle between the horizontal and the line of sight of the object.

Angles of elevation are *always* measured from the *horizontal*.





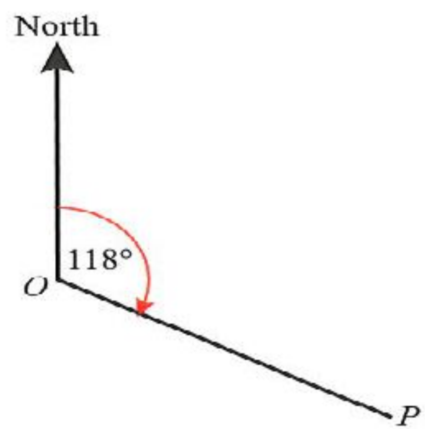
Bearings

You have now used scale drawings to find distances between objects and to measure angles. When you want to move from one position to another, you not only need to know how far you have to travel but you need to know the direction. One way of describing directions is the **bearing**. This description is used around the world.

The angle 118°, shown in the diagram, is measured clockwise from the north direction. Such an angle is called a bearing.

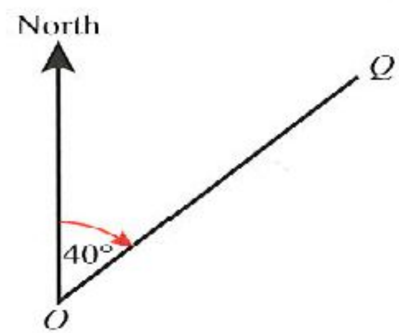
All bearings are measured clockwise from the north direction.

Here the bearing of P from O is 118°.



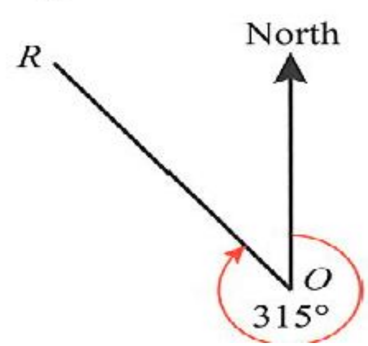
If the angle is less than 100° you still use three figures so that it is clear that you mean to use a bearing.

Here the bearing of Q from O is 040°.



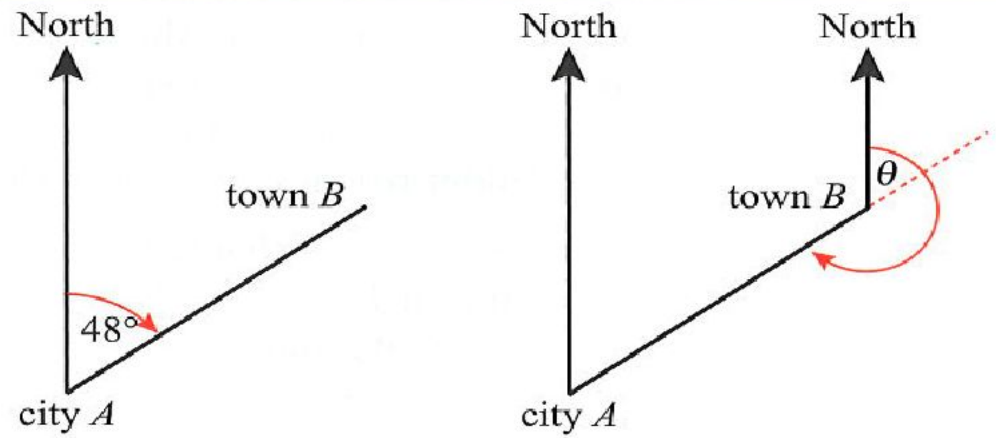
Since you *always* measure clockwise from north it is possible for your bearing to be a reflex angle.

Here the bearing of R from O is 315°.



Worked example 2

The bearing of town B from city A is 048°. What is the bearing of city A from town B?



In the second diagram, the two north lines are parallel. Hence angle $\theta = 48°$ (using the properties of corresponding angles).

The bearing of city A from town B = $48° + 180° = 228°$.

Notice that the difference between the two bearings (48° and 228°) is 180°.

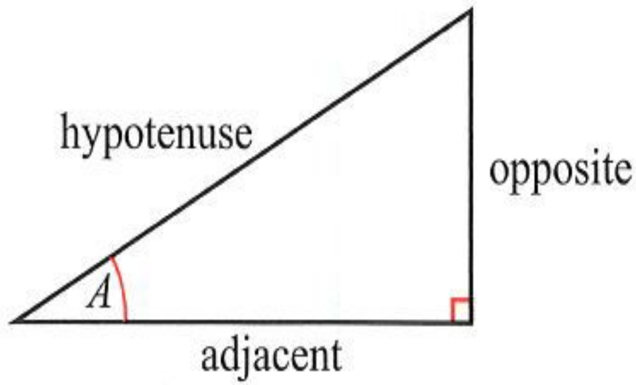
This is an example of a 'back' bearing. If you know the bearing of point X from point Y then, to find the bearing to return to point Y from point X, you add 180° to the original bearing (or subtract 180° if adding would give a value greater than 360°).

Always make sure that you draw a clear diagram and mark all north lines clearly.

Understanding the tangent, cosine and sine ratios

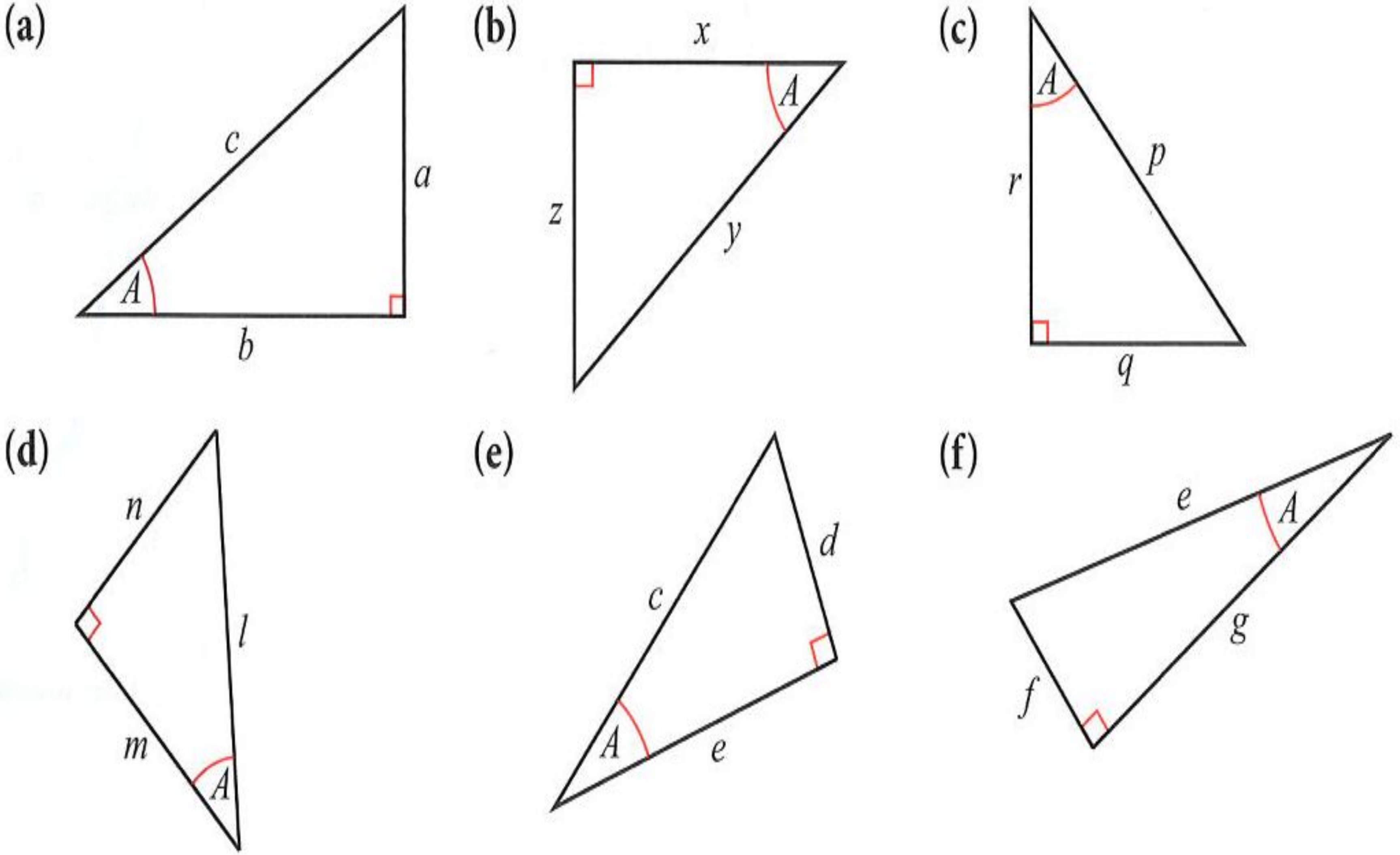
Labelling the sides of a right-angled triangle

You will have already learned that the longest side of a right-angled triangle is called the **hypotenuse**. If you take one of the two non right-angles in the triangle for reference then you can also 'name' the two shorter sides:



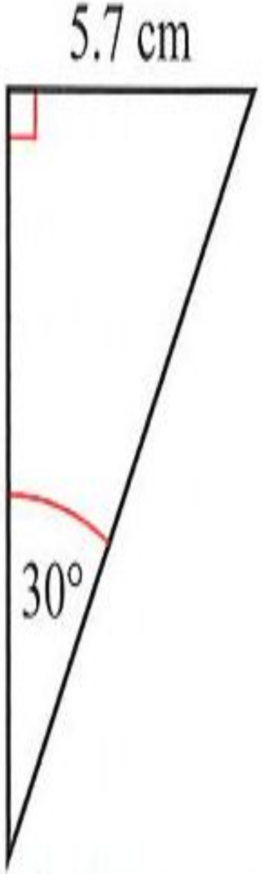
Notice that the **adjacent** is the side of the triangle that touches the angle A, but is not the hypotenuse. The third side does not meet with angle A at all and is known as the **opposite.** Throughout the remainder of the chapter, opp(A) will be used to mean the length of the opposite side, and adj(A) to mean the length of the adjacent. The hypotenuse does not depend upon the position of angle A, so is just written as 'hypotenuse' (or hyp).

1 For each of the following triangles write down the letters that correspond to the length of the hypotenuse and the values of opp(A) and adj(A).



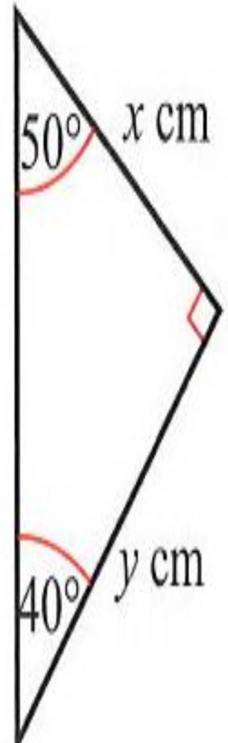
2 In each case copy and complete the statement written underneath the triangle.

(a)



opp(30°) = ... cm

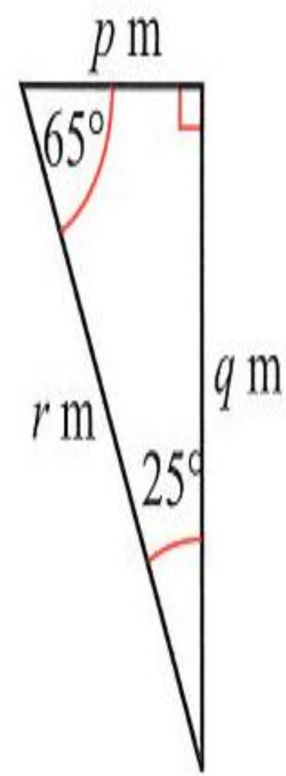
(b)



opp(40°) = ... cm

adj(50°) = ... cm

(c)



.......(65°) = q m

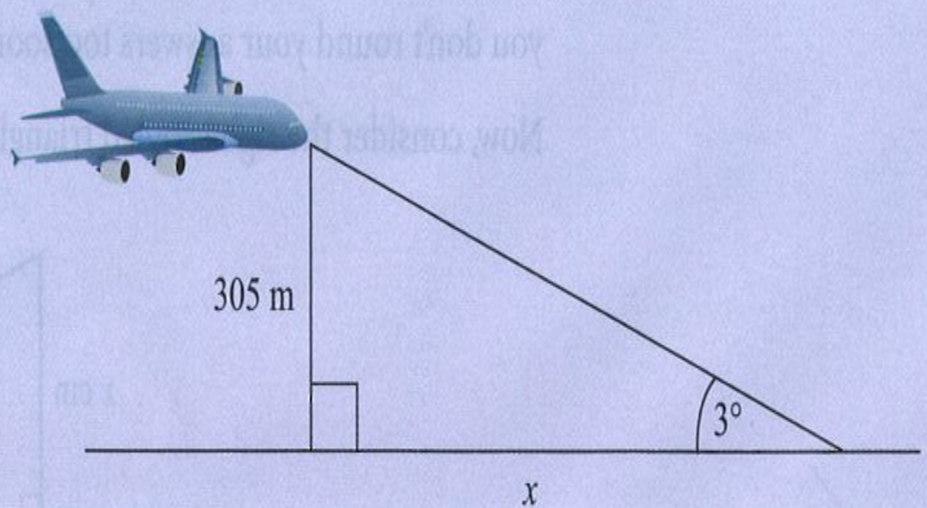
.......(25°) = p m

.... = r m

Investigation

Worked example 5

The angle of approach of an airliner should be 3°. If a plane is 305 metres above the ground, how far should it be from the airfield?



$$\tan 3^\circ = \frac{305}{x}$$

$$\Rightarrow x \tan 3^\circ = 305$$

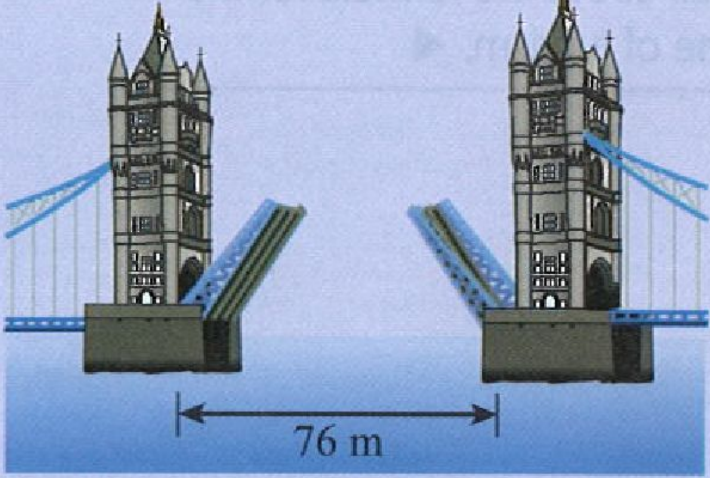
$$\Rightarrow x = \frac{305}{\tan 3^\circ}$$

$$= 5819.74\ldots$$

$$\approx 5820 \text{ (nearest metre)}$$

Worked example 12

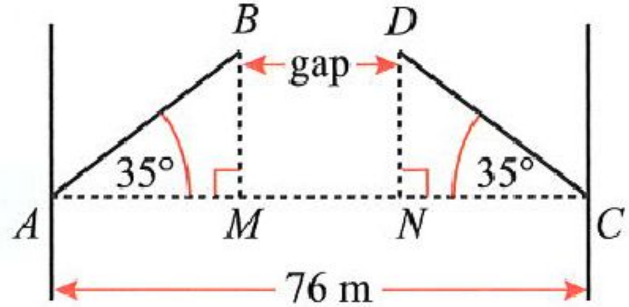
The span between the towers of Tower Bridge in London is 76 m. When the arms of the bridge are raised to an angle of 35°, how wide is the gap between their ends?



Tip

Give your answer to 3 significant figures. You will need to do this in the exam if no degree of accuracy is specified.

Here is a simplified labelled drawing of the bridge, showing the two halves raised to 35°.



The gap $= BD = MN$ and $MN = AC - (AM + NC)$.

The right-angled triangles ABM and CDN are congruent, so $AM = NC$.
When the two halves are lowered, they must meet in the middle.

$$\therefore AB = CD = \frac{76}{2} = 38\,\text{m}$$

$$\text{In } \Delta ABM,\ \cos 35^\circ = \frac{\text{adj}(35^\circ)}{\text{hyp}} = \frac{AM}{38}$$

$$AM = 38 \times \cos 35^\circ$$
$$= 31.1277\ldots\text{m}$$

$$\therefore MN = 76 - (31.1277\ \ldots + 31.1277\ \ldots)$$
$$= 13.744\ \ldots\text{m}$$

The gap $BD = 13.7\,\text{m}$ (to 3sf)

Angles between 90° and 180°

If θ is obtuse, then:

$\sin\theta = +\sin(180° - \theta)$

$\cos\theta = -\cos(180° - \theta)$ (This is the same as $-\cos\theta = \cos(180° - \theta)$).

In practice, you would use a calculator to find the trig ratios. The rules for obtuse angles are programmed into all scientific calculators.

Use your calculator to check that:

$\sin 100° = +0.984807753 = \sin 80°$

$\cos 100° = -0.173648177 = -\cos 80°$

$\sin 150° = +0.5 = \sin 30°$

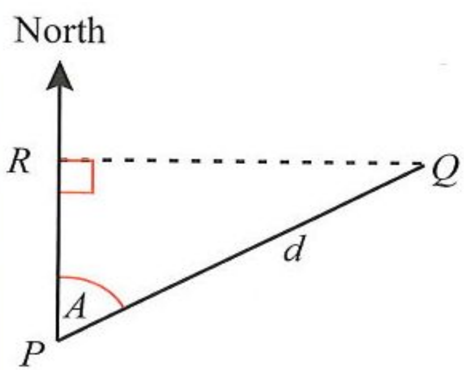
$\cos 150° = -0.866025403 = -\cos 30°$

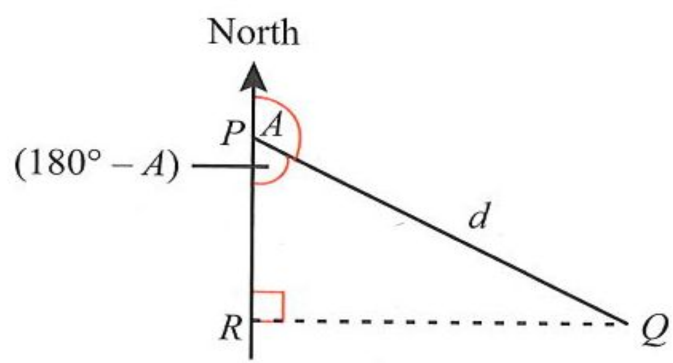
In effect, this means that an angle and its supplement have the same sine value. For the cosine of an angle and its supplement, the signs will differ.

Worked example 13

Village Q is D kilometres from village P on a bearing of $A°$. How far east and how far north is Q from P?

If A is acute, then you work out $\sin A$ and $\cos A$.





From the diagrams, you can see that:

$\sin A = \dfrac{RQ}{d}$ and $\cos A = \dfrac{PR}{d}$

$\therefore\ RQ = d \sin A$ and $PR = d \cos A$

Now suppose A is obtuse.

$QPR = 180° - A$

$\therefore\ \sin(180° - A) = \dfrac{RQ}{d}$ and $\cos(180° - A) = \dfrac{PR}{d}$

$\therefore\ RQ = d \sin(180° - A)$ and $PR = d \cos(180° - A)$

Worked example 14

Which acute angle has the same sine as 120°?

$\sin(180° - \theta) = \sin\theta$ But in this case, $\theta = 120°$

$180° - \theta = 120$

$180° - 120° = \theta$

$60° = \theta$

$\sin 60° = \sin 120°$

Worked example 15

Express each of the following in terms of another angle between 0° and 180°.

(a) $\cos 100°$ **(b)** $-\cos 35°$

(a) $\cos(180° - \theta) = -\cos\theta$ In this case $\theta = 100°$

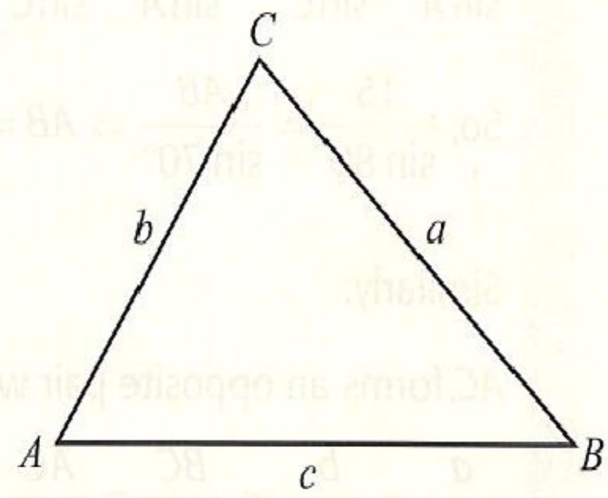
$\cos 100° = -\cos(180° - 100°) = -\cos 80°$

(b) $-\cos\theta = \cos(180° - \theta)$

$-\cos 35° = \cos 145°$

The sine and cosine rules

The sine and cosine ratios are not only useful for right-angled triangles. To understand the following rules you must first look at the standard way of labelling the angles and sides of a triangle. Look at the triangle shown in the diagram.



Notice that the sides are labelled with lower case letters and the angles are labelled with upper case letters. The side that is placed opposite angle A is labelled 'a', the side that is placed opposite angle B is labelled 'b' and so on.

The sine rule

For the triangle shown above, the following are true:

$$\frac{\sin A}{a} = \frac{\sin B}{b} \quad \text{and} \quad \frac{\sin A}{a} = \frac{\sin C}{c} \quad \text{and} \quad \frac{\sin B}{b} = \frac{\sin C}{c}$$

These relationships are usually expressed in one go:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

This is the **sine rule**. This version of the rule, with the sine ratios placed on the tops of the fractions, is normally used to calculate angles.

The formulae can also be turned upside down when you want to calculate lengths:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

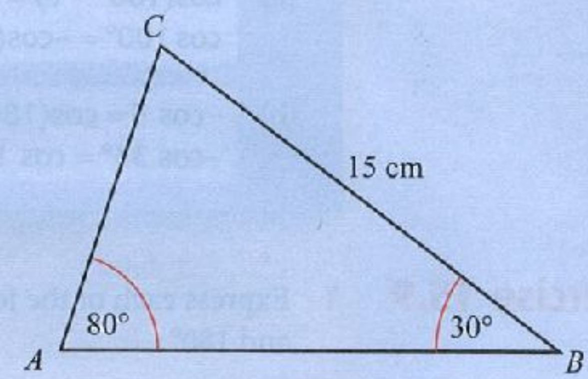
You should remember that this represents *three* possible relationships.

Notice that in each case, both the upper and lower case form of each letter is used. This means that each fraction that you use requires an angle and the length of its opposite side.

Worked example 16

In $\triangle ABC$, $\hat{A} = 80°$, $\hat{B} = 30°$ and side $BC = 15$ cm.

Calculate the size of $\hat{C}$ and the lengths of the sides AB and AC.



To calculate the angle C, use the fact that the sum of the three angles in a triangle is always 180°.

So, $C + 80 + 30 = 180 \Rightarrow C = 180 - 30 - 80 = 70°$

Now think about the side AB. AB is opposite the angle C (forming an 'opposite pair') and side BC is opposite angle A, forming a second 'opposite pair'.

So, write down the version of the sine rule that uses these pairs:

$$\frac{a}{\sin A} = \frac{c}{\sin C} \Rightarrow \frac{BC}{\sin A} = \frac{AB}{\sin C}$$

So, $\frac{15}{\sin 80°} = \frac{AB}{\sin 70°} \Rightarrow AB = \frac{15}{\sin 80°} \times \sin 70° = 14.3\,\text{cm}\,(3\text{sf})$

Similarly:

AC forms an opposite pair with angle B, so once again use the pair BC and angle A:

$$\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \frac{BC}{\sin A} = \frac{AC}{\sin B}$$

So, $\frac{15}{\sin 80°} = \frac{AC}{\sin 30°} \Rightarrow AC = \frac{15}{\sin 80°} \times \sin 30° = 7.62\,\text{cm}\,(3\text{sf})$

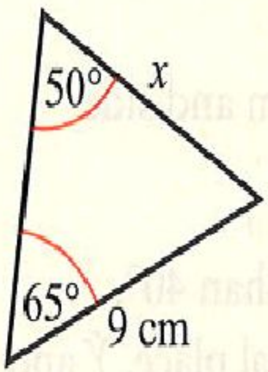
The ambiguous case of the sine rule

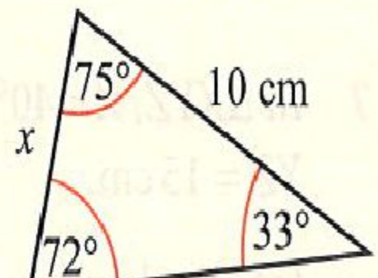
The special properties of the sine function can lead to more than one possible answer. The following example demonstrates how this may happen.

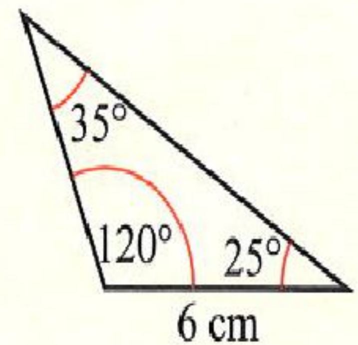
1 Find the value of x in each of the following equations.

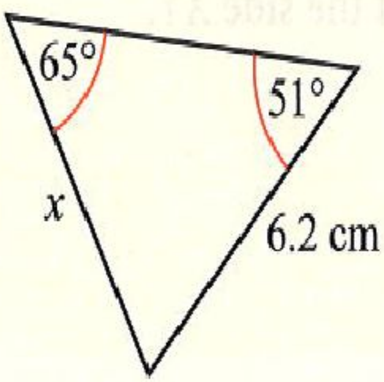
(a) $\dfrac{x}{\sin 50} = \dfrac{9}{\sin 38}$ (b) $\dfrac{x}{\sin 25} = \dfrac{20}{\sin 100}$ (c) $\dfrac{20.6}{\sin 50} = \dfrac{x}{\sin 70}$ (d) $\dfrac{\sin x}{11.4} = \dfrac{\sin 63}{16.2}$

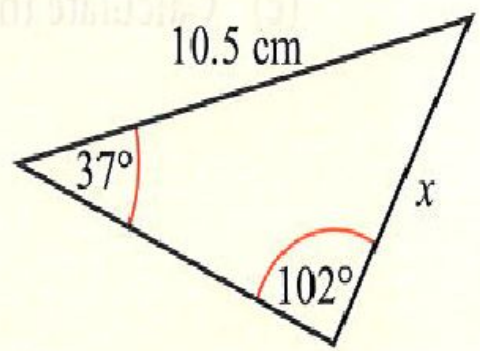
2 Find the length of the side marked x in each of the following triangles.

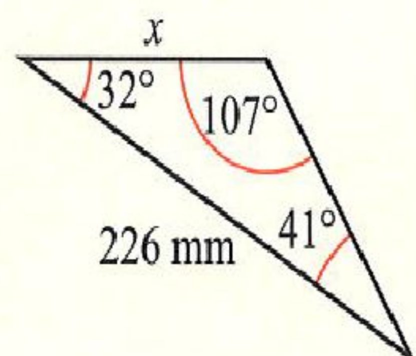






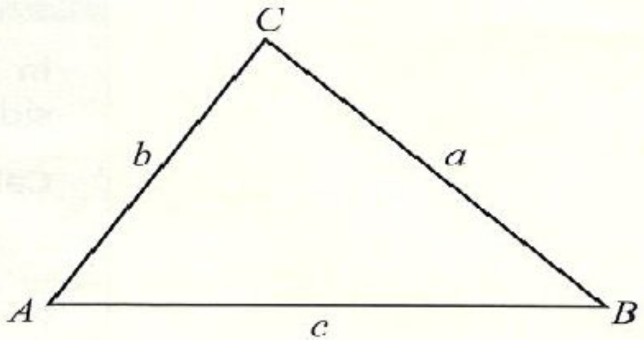






Cosine rule

For the **cosine rule**, consider a triangle labelled in exactly the same way as that used for the sine rule.



The cosine rule is stated as a single formula:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

Notice the all three sides are used in the formula, and just one angle. The side whose square is the subject of the formula is opposite the angle used (hence they have the same letter but in a different case). This form of the cosine rule is used to find unknown sides.

By rearranging the labels of angles (but making sure that opposite sides are still given the lower case version of the same letter for any given angle) the cosine rule can be stated in two more possible ways:

$$b^2 = a^2 + c^2 - 2ac\cos B \qquad \text{or} \qquad c^2 = a^2 + b^2 - 2ab\cos C$$

Notice, also, that you can take any version of the formula to make the cosine ratio the subject.

This version can be used to calculate angles:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$\Rightarrow a^2 + 2bc\cos A = b^2 + c^2$$

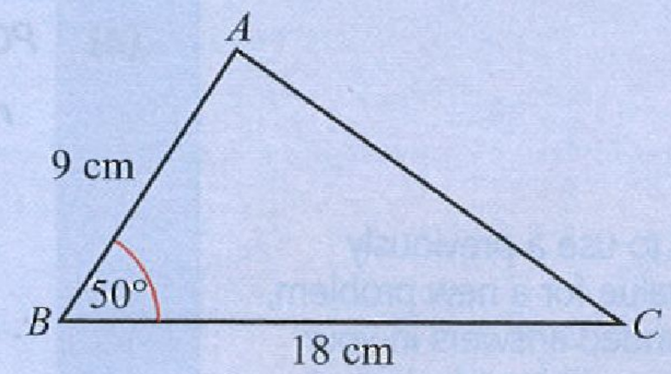
$$\Rightarrow 2bc\cos A = b^2 + c^2 - a^2$$

$$\Rightarrow \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Worked example 18

In ΔABC, $\hat{B} = 50°$, side $AB = 9$ cm and side $BC = 18$ cm.

Calculate the length of AC.



Notice that $AC = b$ and you know that $\hat{B} = 50°$.

Use the cosine rule in the form, $b^2 = a^2 + c^2 - 2ac\cos B$

$$b^2 = 9^2 + 18^2 - (2 \times 9 \times 18 \times \cos 50°)$$

$$= 81 + 324 - (208.2631\ldots)$$

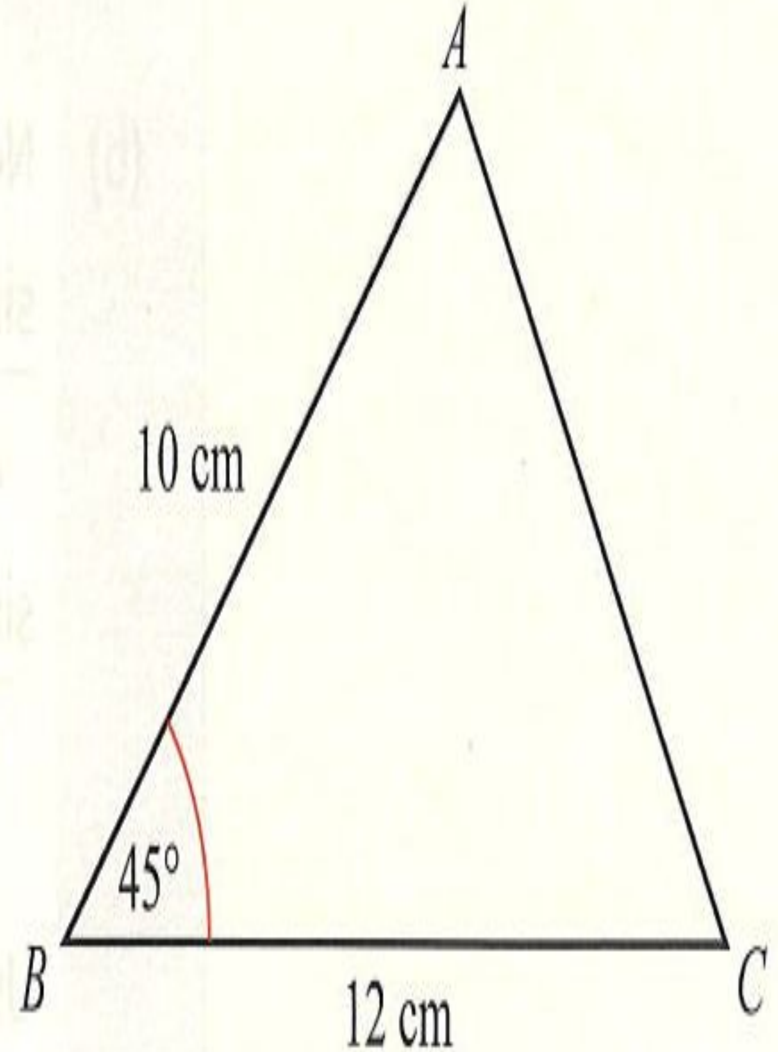
$$= 196.7368\ldots$$

$$\therefore b = \sqrt{196.7368\ldots}$$

$$= 14.0262\ldots$$

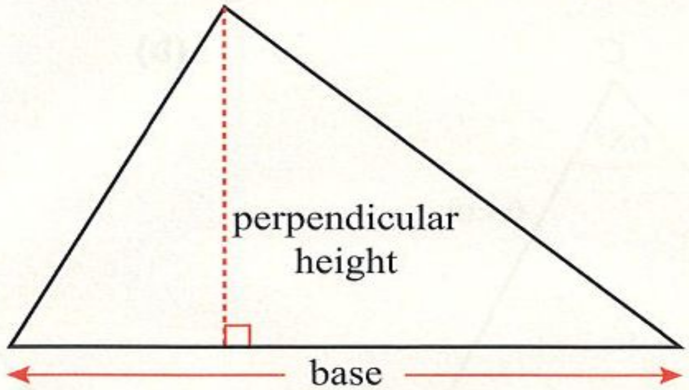
Length of $AC = 14.0$ cm (to 3 s.f.)

1 In ΔABC, $\hat{B} = 45°$, side $AB = 10$ cm and side $BC = 12$ cm. Calculate the length of side AC.



Area of a triangle

You already know that the area of a triangle is given by the following formula:



$$\text{Area} = \frac{1}{2} \times \text{base} \times \text{perpendicular height}$$

This method can be used if you know both the length of the base and perpendicular height but if you don't have these values you need to use another method.

In fact you could use any side of the triangle as the base and draw the perpendicular height accordingly. This means that the area can also be calculated with:

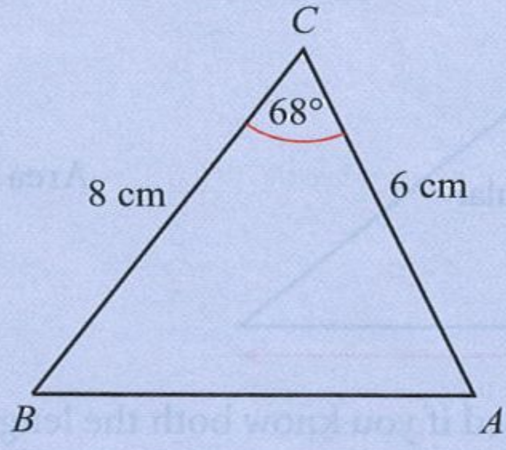
$$\text{Area} = \frac{1}{2} ac \sin B \qquad \text{or} \qquad \text{Area} = \frac{1}{2} bc \sin A$$

In each case the sides used meet at the angle that has been included.

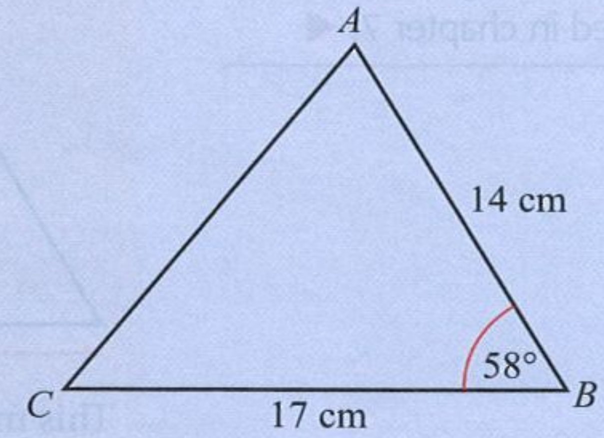
Worked example 22

Calculate the areas of each of the following shapes.

(a)



(b)



(a)

$$\text{Area} = \frac{1}{2}ab\sin C$$

$$= \frac{1}{2} \times 8 \times 6 \times \sin 68°$$

$$= 22.3\,\text{cm}^2 \text{ (to 1dp)}$$

(b)

$$\text{Area} = \frac{1}{2}ac\sin B$$

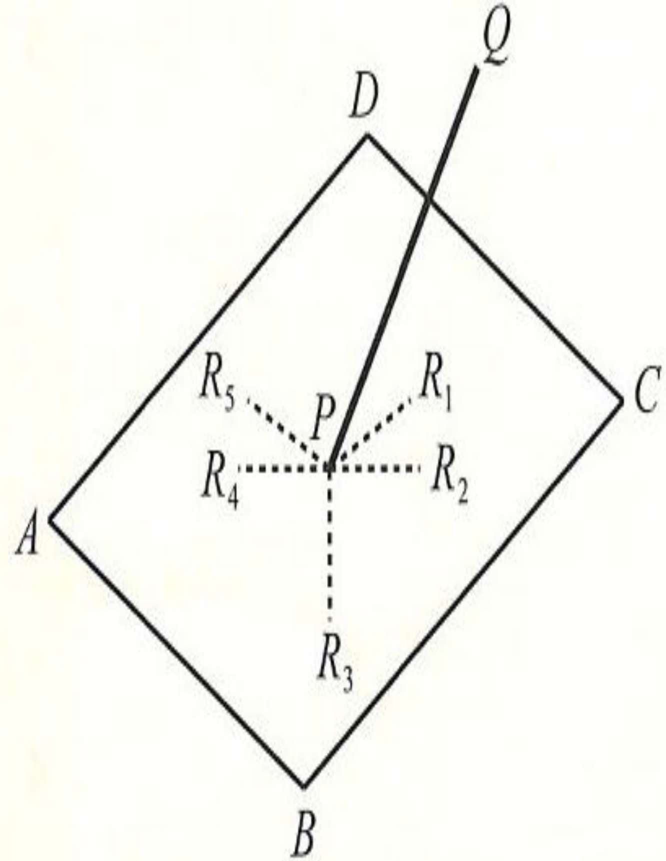
$$= \frac{1}{2} \times 17 \times 14 \times \sin 58°$$

$$= 100.9\,\text{cm}^2 \text{ (to 1dp)}$$

Trigonometry in three dimensions

When you work with solids you may need to calculate the angle between an edge, or a diagonal, and one of the faces. This is called the angle between a line and a plane.

Consider a line PQ, which meets a plane $ABCD$ at point P. Through P draw lines PR_1, PR_2, PR_3, … in the plane and consider the angles QPR_1, QPR_2, QPR_3…

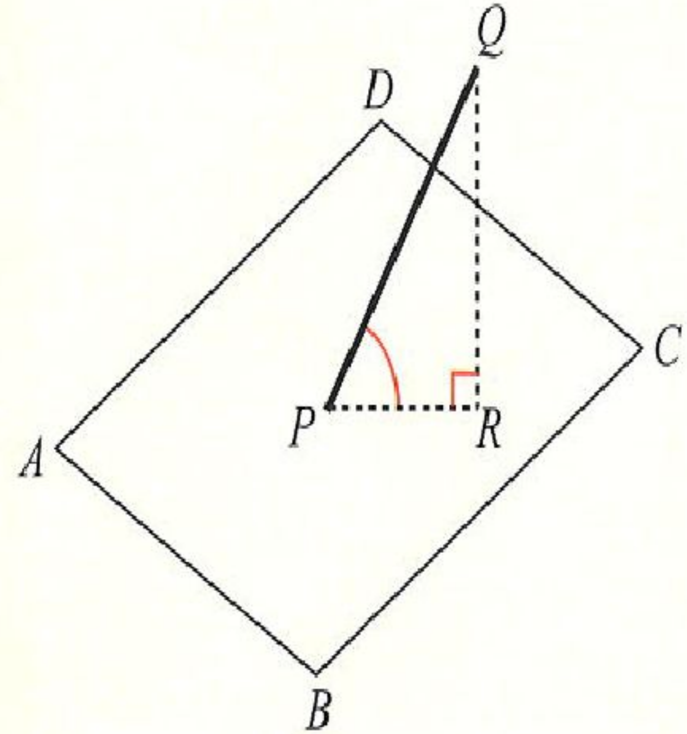


- If PQ is perpendicular to the plane, all these angles will be right angles.
- If PQ is not perpendicular to the plane, these angles will vary in size.

It is the smallest of these angles which is called the angle between the line PQ and the plane $ABCD$.

To identify this angle, do the following:

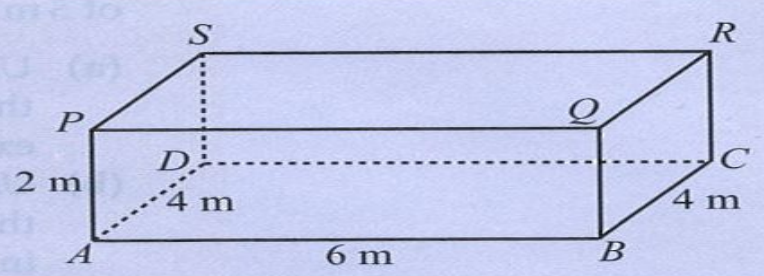
- From Q draw a perpendicular to the plane. Call the foot of this perpendicular R.
- The angle between the line PQ and the plane is $Q\hat{P}R$.



PR is called the **projection** of PQ on the plane $ABCD$.

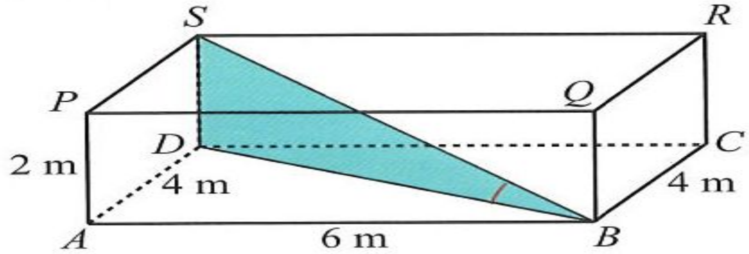
Worked example 24

The diagram represents a room which has the shape of a cuboid. $AB = 6\,\text{m}$, $AD = 4\,\text{m}$, and $AP = 2\,\text{m}$. Calculate the angle between the diagonal BS and the floor $ABCD$.



First identify the angle required. B is the point where the diagonal BS meets the plane $ABCD$.

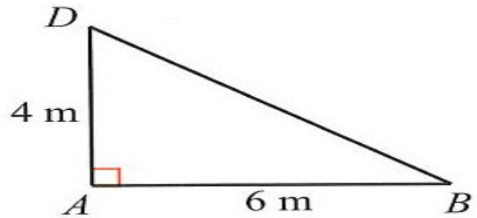
SD is the perpendicular from S to the plane $ABCD$ and so DB is the projection of SB onto the plane.



The angle required is $S\hat{B}D$.

You know that $\triangle SBD$ has a right angle at D and that $SD = 2\,\text{m}$ (equal to AP).

To find $S\hat{B}D$, you need to know the length of DB or the length of SB. You can find the length of BD by using Pythagoras' theorem in $\triangle ABD$.



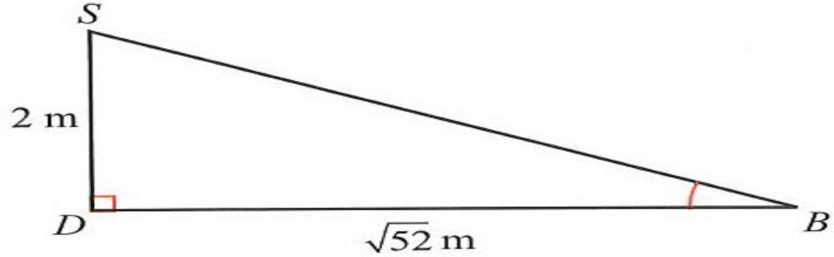
$$BD^2 = 6^2 + 4^2 = 36 + 16 = 52$$

$$BD = \sqrt{52}$$

So, using right-angled triangle SBD:

$$\tan B = \frac{\text{opp}(B)}{\text{adj}(B)} = \frac{SD}{BD} = \frac{2}{\sqrt{52}}$$

$$S\hat{B}D = \tan^{-1}\left(\frac{2}{\sqrt{52}}\right) = 15.5013\ldots$$



The angle between diagonal BS and the floor $ABCD = 15.5°$ (to 1 d.p.)

Summary

Do you know the following?

- A scale drawing is an accurate diagram to represent something that is much bigger, or much smaller.
- An angle of elevation is measured upwards from the horizontal.
- An angle of depression is measured downwards from the horizontal.
- Bearings are measured clockwise from north.
- The ratio of any two lengths in a right-angled triangle depends on the angles in the triangle:
  - $\sin A = \dfrac{\text{opp}(A)}{\text{hyp}}$.
  - $\cos A = \dfrac{\text{adj}(A)}{\text{hyp}}$.
  - $\tan A = \dfrac{\text{opp}(A)}{\text{adj}}$.
- You can use these trigonometric ratios to calculate an unknown angle from two known sides.
- You can use these trigonometric ratios to calculate an unknown side from a known side and a known angle.

- The sine, cosine and tangent function can be extended beyond the angles in triangles.
- The sine and cosine rules can be used to calculate unknown sides and angles in triangles that are not right-angled.
- The sine rule is used for calculating an angle from another angle and two sides, or a side from another side and two known angles. The sides and angles must be arranged in opposite pairs.
- The cosine rule is used for calculating an angle from three known sides, or a side from a known angle and two known sides.
- You can calculate the area of a non right-angled triangle by using the sine ratio.

Are you able to.....?

- calculate angles of elevation
- calculate angles of depression
- use trigonometry to calculate bearings
- identify which sides are the opposite, adjacent and hypotenuse
- calculate the sine, cosine and tangent ratio when given lengths in a right-angled triangle
- use the sine, cosine and tangent ratios to find unknown angles and sides
- solve more complex problems by extracting right-angled triangles and combining sine, cosine and tangent ratios
- use the sine and cosine rules to find unknown angles and sides in right-angled triangles
- use sine and cosine rules to find unknown angles and sides in triangles that are not right-angled
- use trigonometry in three dimensions
- find the area of a triangle that is not right-angled.