Complexity and Fragility?

Or, what color is your herring?

John Doyle
Control and Dynamical Systems
BioEngineering
Electrical Engineering
Caltech

with Prajna, Papachristodoulou, and Parrilo

The popular media has given us three similar visions of our technological future


IS THERE
ANY HOPE?

What is the ultimate showstopper?

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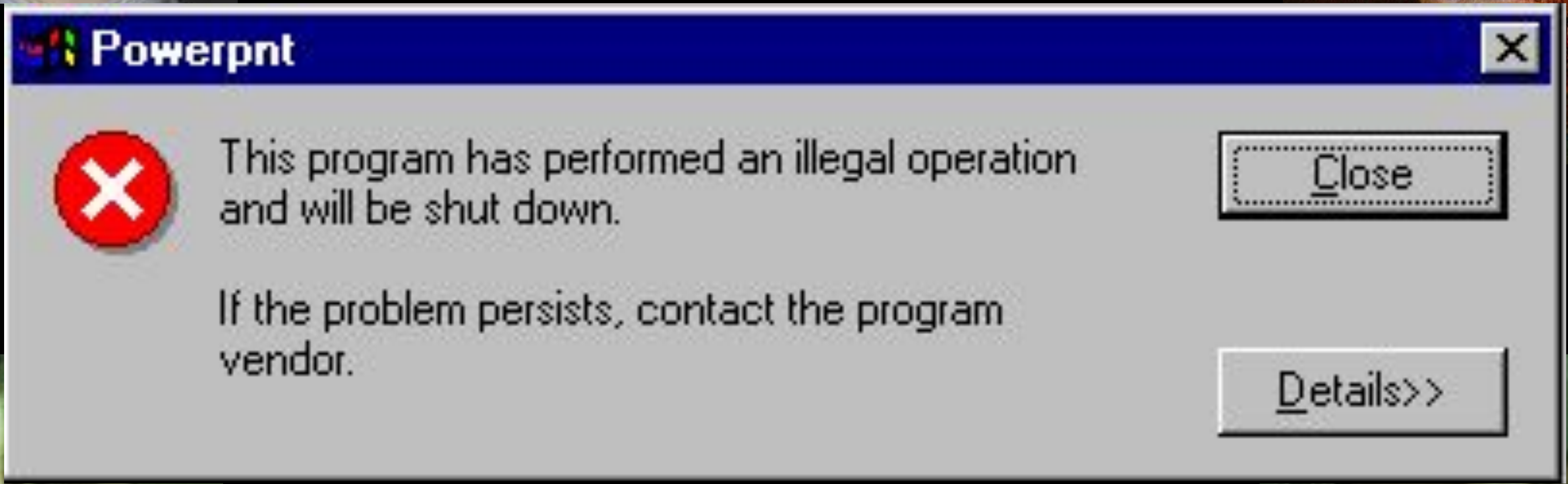
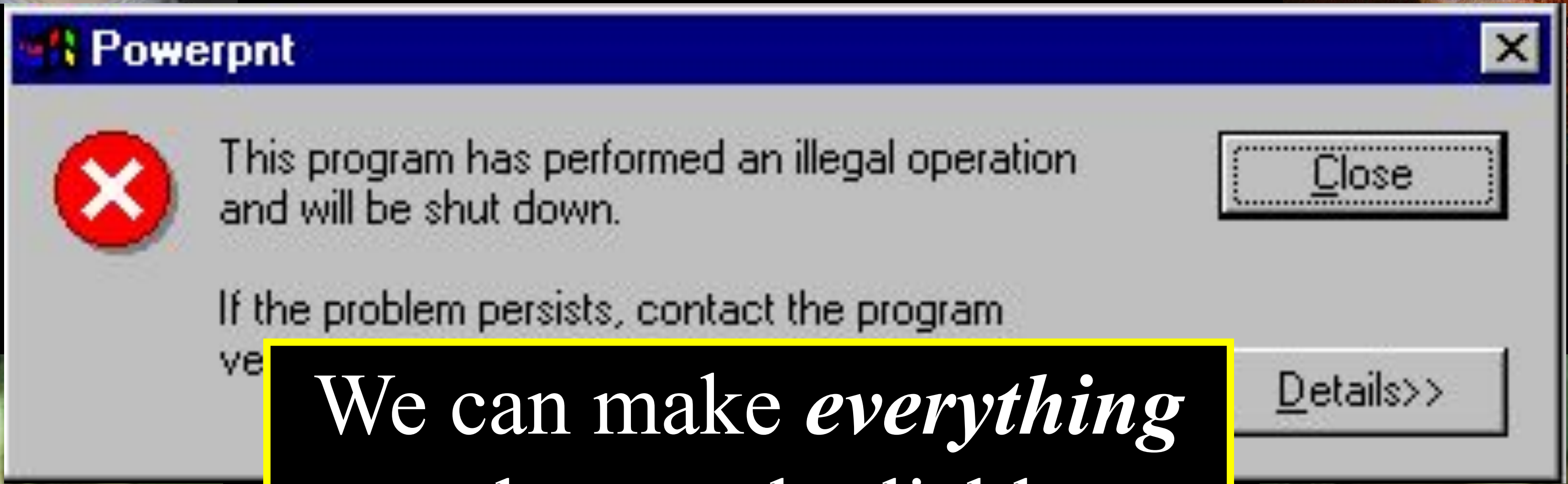


IS THERE ANY HOPE?

What is the ultimate showstopper?



We can make *everything* as robust and reliable as our software!

ANY HOPE?

Engineering design objectives

1. **Robust** to uncertainty in environment and components
2. **Efficient** use of scarce resources
3. **Scalable** to large system sizes

(to do this, it may be necessary to have high *internal* complexity (***complicated***),

but we want simple, robust, verifiable external behavior, so…)

Engineering design objectives

1. **Robust**
2. **Efficient**
3. **Scalable**

4. **Verifiable** with short proofs

Bottom line

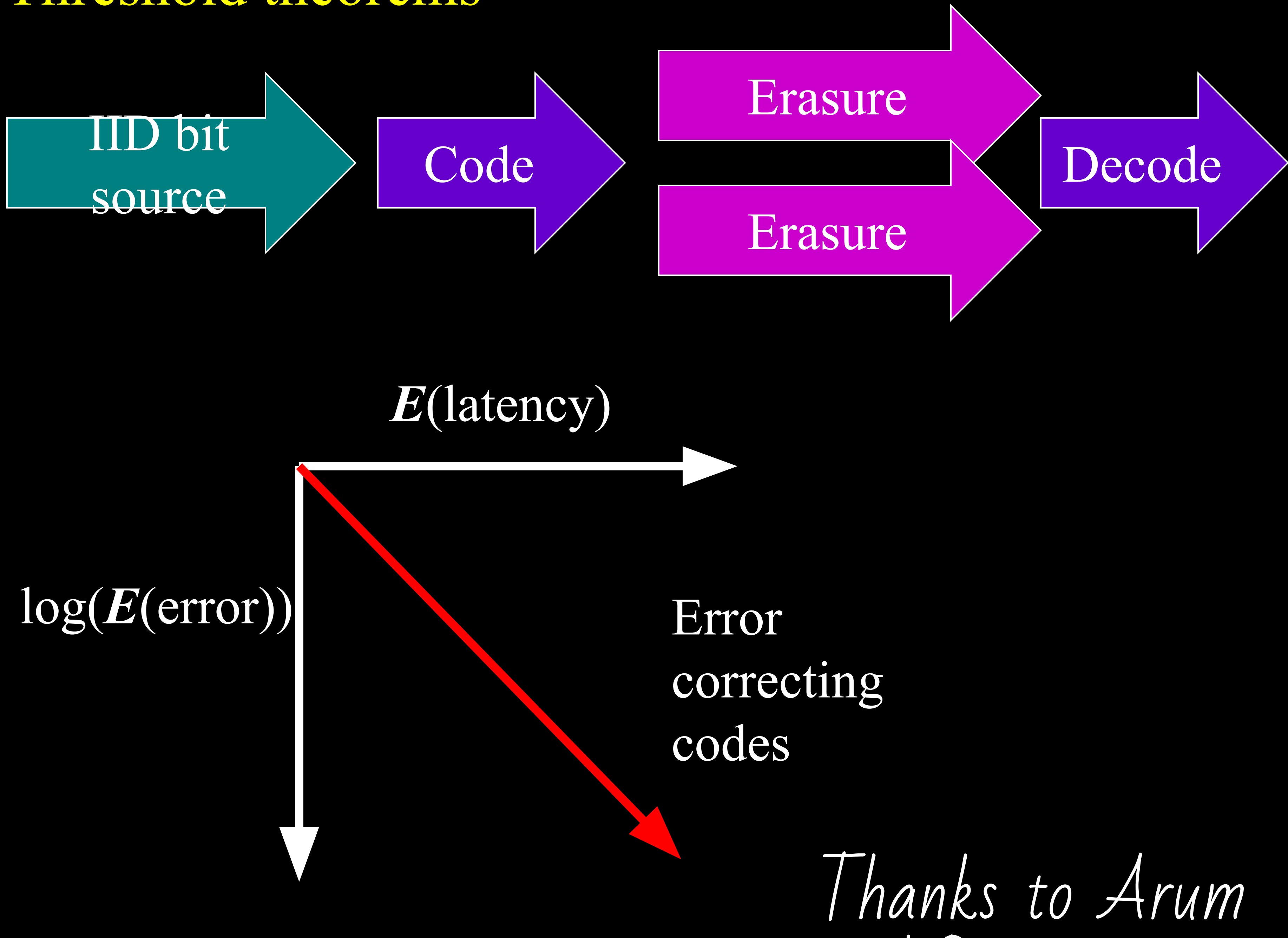
- Want robustness & efficiency that is ***verifiably*** so
- May require highly complex organization and structure

Robustness, evolvability, and verifiability are compatible

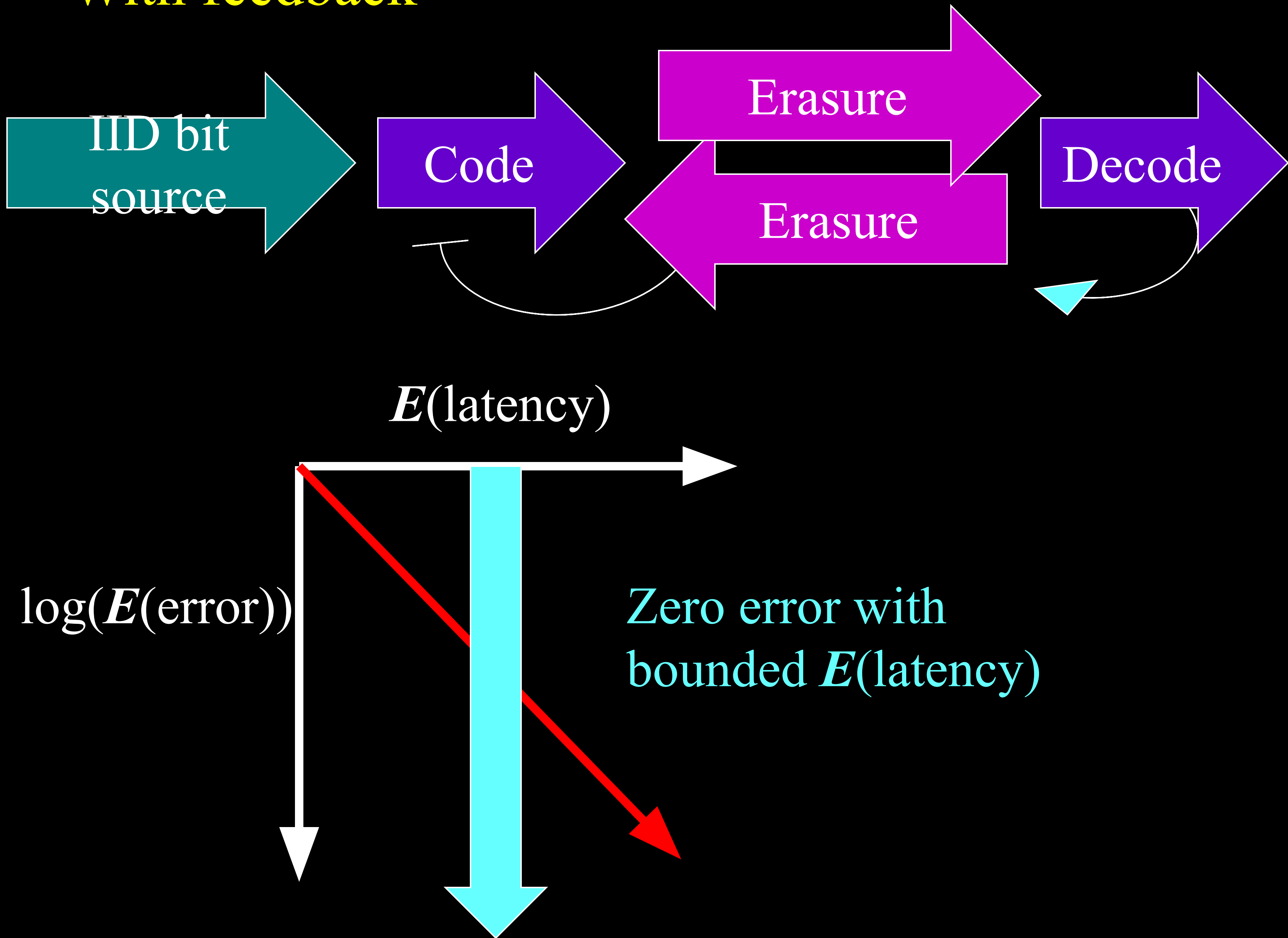
and

Tradeoff to some extent against efficiency, cost, complexity, etc.

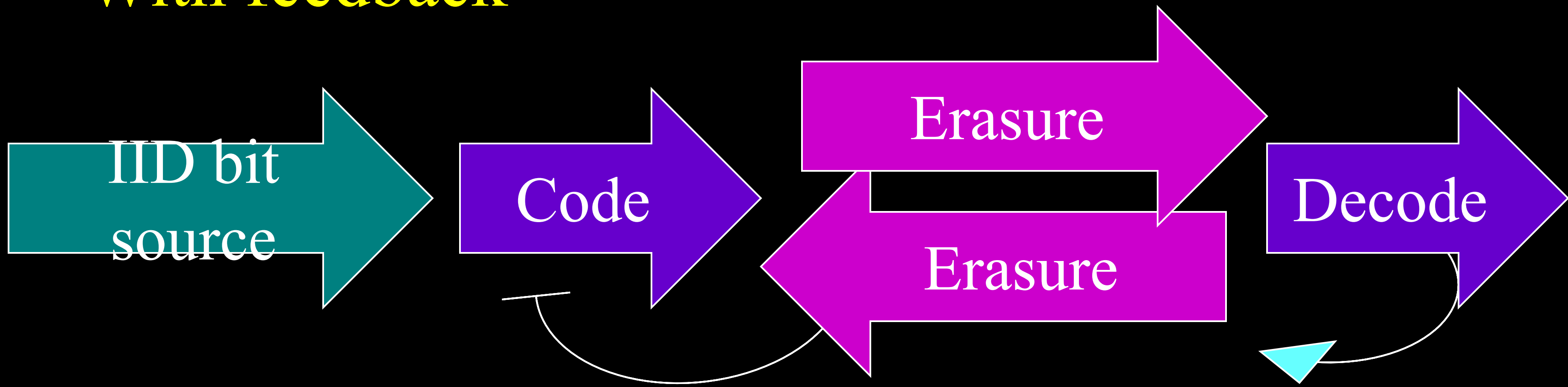
Threshold theorems



With feedback



With feedback



Zero error with bounded ***E***(latency)!

Danger: new fragilities!

Two *opposite* views of complexity

Physics:

- Pattern formation by reaction/diffusion
- Edge-of-chaos
- Order for free
- Self-organized criticality
- Phase transitions
- Scale-free networks
- Equilibrium, linear
- Nonlinearity & complexity as exotica

Engineering:

- Constraints
- Tradeoffs
- Structure
- Organization
- Optimality
- Robustness/fragility
- Verification
- Far from equilibrium
- Nonlinearity & complexity as tool

Two *opposite* views of life

Physics:

- If you are dead, you are likely to stay that way

Engineering:

- If you are alive, it is very easy to kill you

The bad news (unfortunately):
Robustness is less fungible with other features than you think.

The good news (hopefully):
If we can identify our fragilities, we can

- verify that we are otherwise robust
- and keep ourselves that way

The "simplest" hard problem

NPP
(Number partitioning problem)

Given $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$

Compute

$$\min_{x_i^2=1} \left|\sum a_i x_i\right|$$

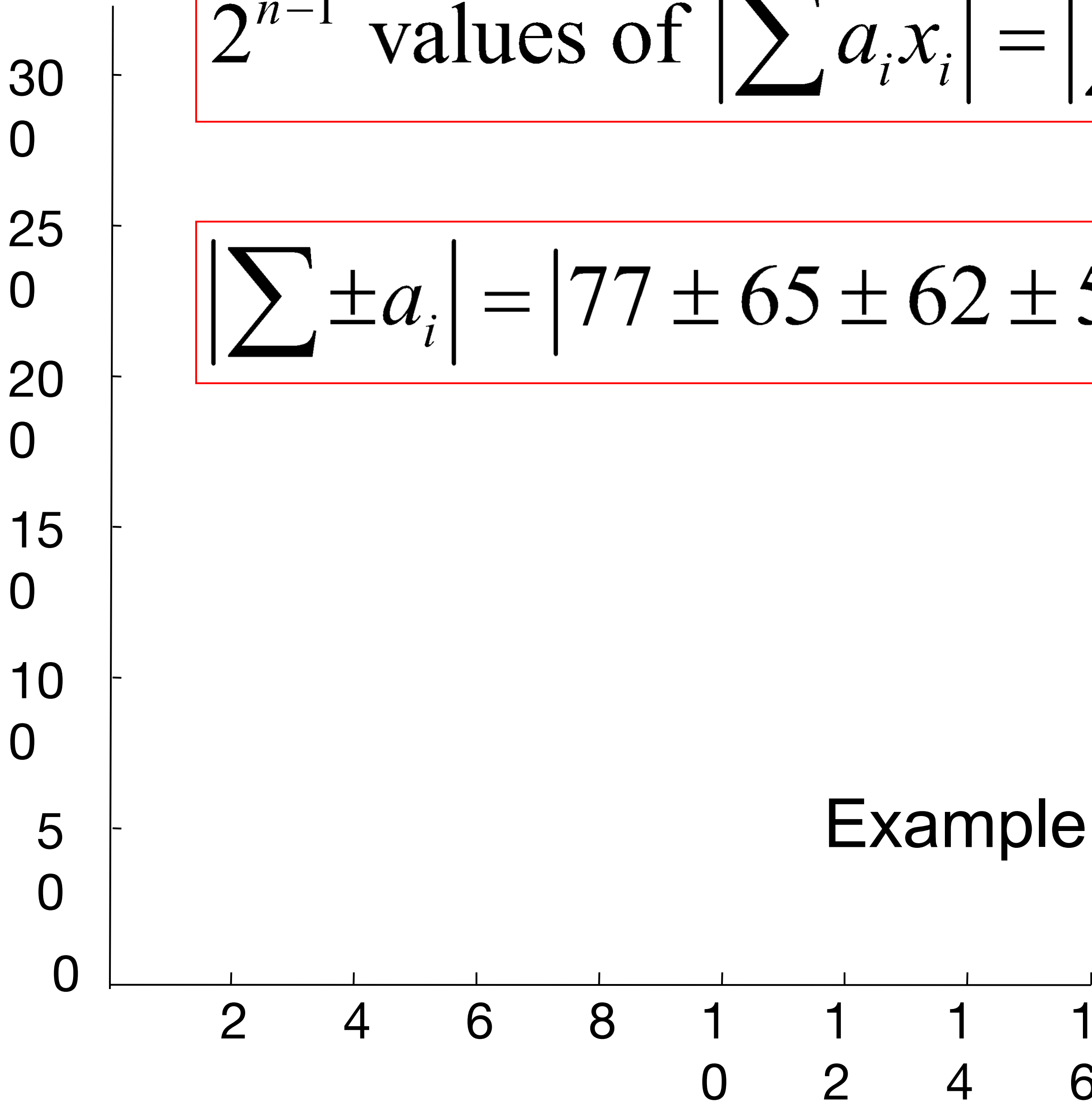
$$= \min_{x_i=\pm 1} \left|\sum a_i x_i\right|$$

$$= \min_{\pm} \left|a_1 \pm a_2 \pm \cdots \pm a_n\right|$$

A "classic" NP complete problem

$$2^{n-1} \text{ values of } \left|\sum a_i x_i\right| = \left|\sum \pm a_i\right|$$

$$\left|\sum \pm a_i\right| = \left|77 \pm 65 \pm 62 \pm 59 \pm 31\right|$$



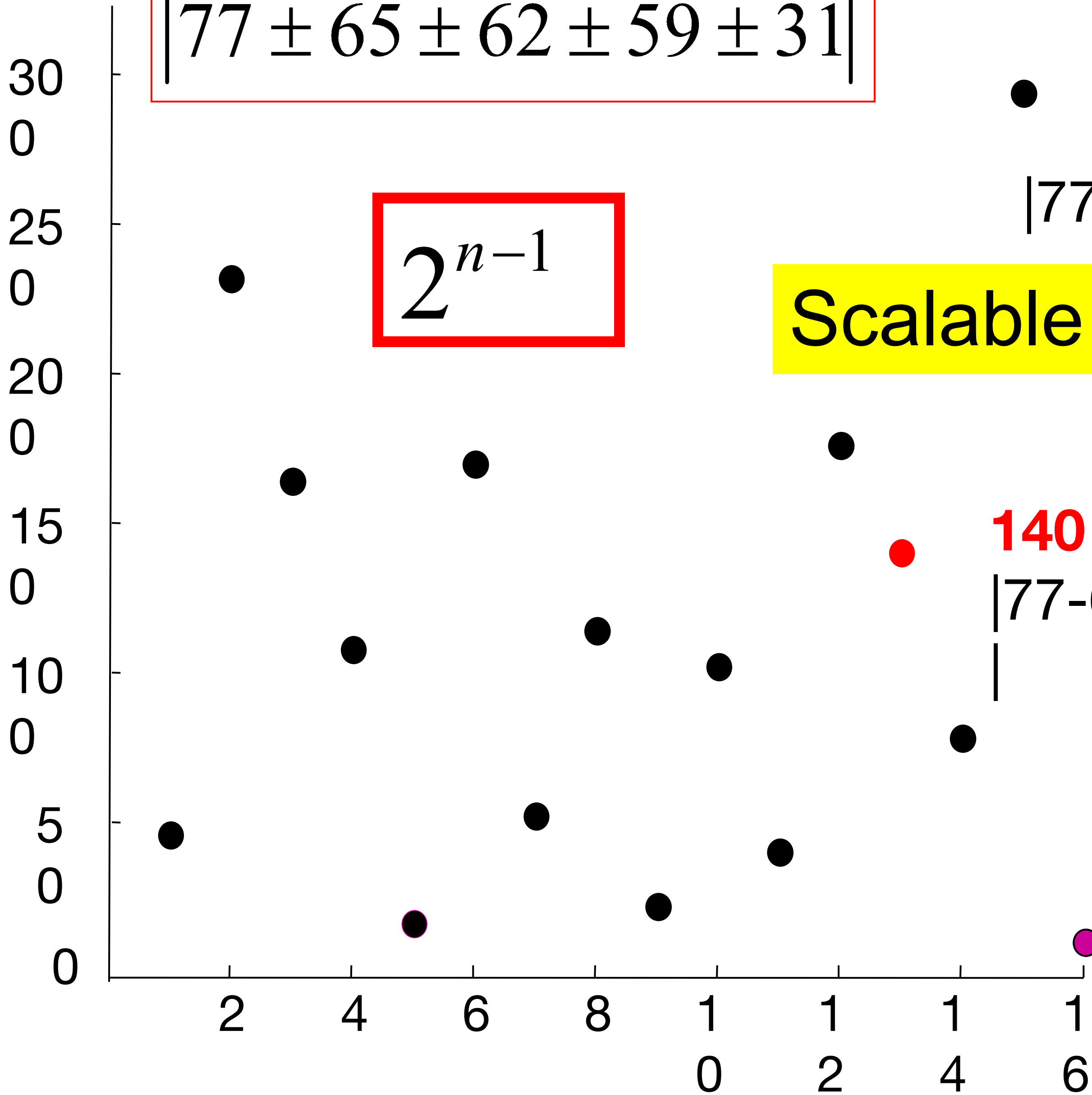
Example

$$a = \begin{bmatrix} 77 \\ 65 \\ 62 \\ 59 \\ 31 \end{bmatrix}$$

$$\left|77 \pm 65 \pm 62 \pm 59 \pm 31\right|$$

$$\boxed{2^{n-1}}$$

Scalable algorithms?

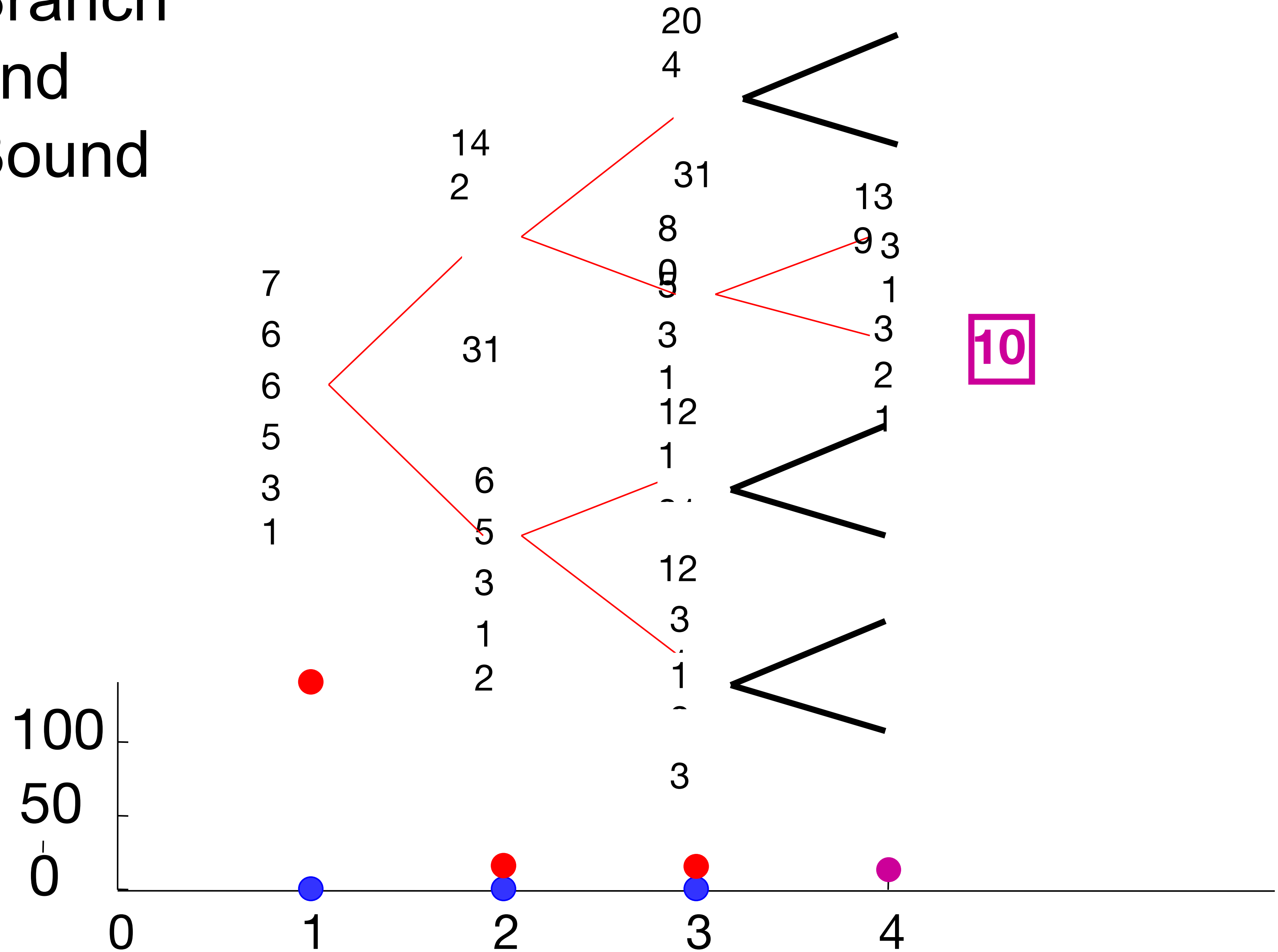


max=294 =
|77+65+62+59+31|

140 =
|77-65-62-59-31|

|77+65-62-59-31|
= min=10

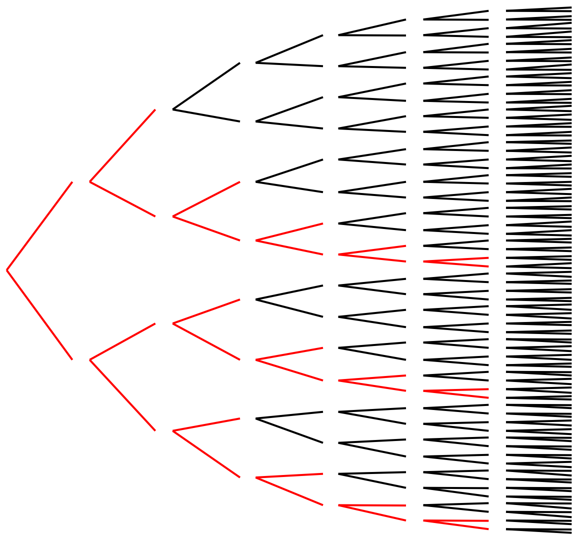
Branch and Bound

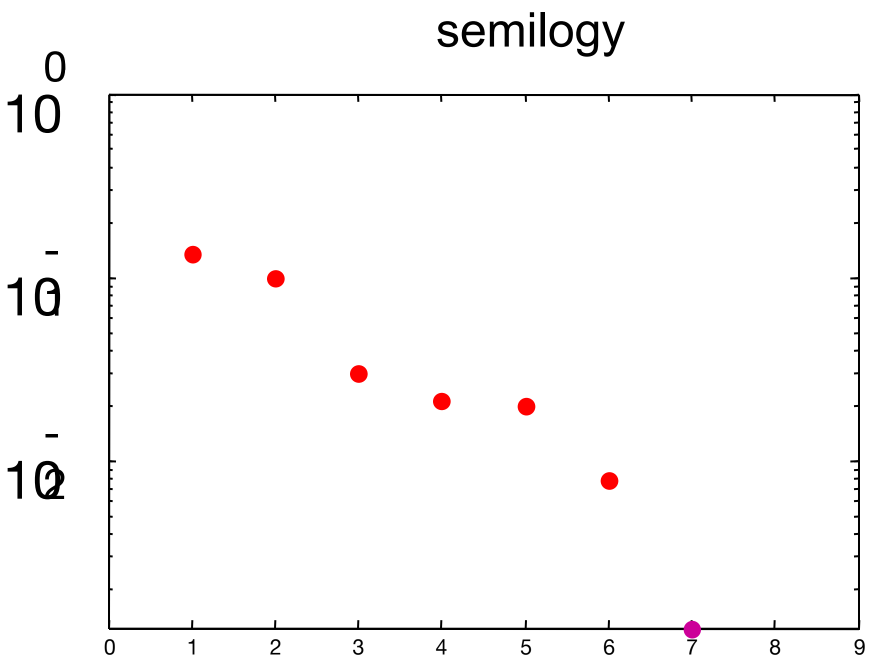


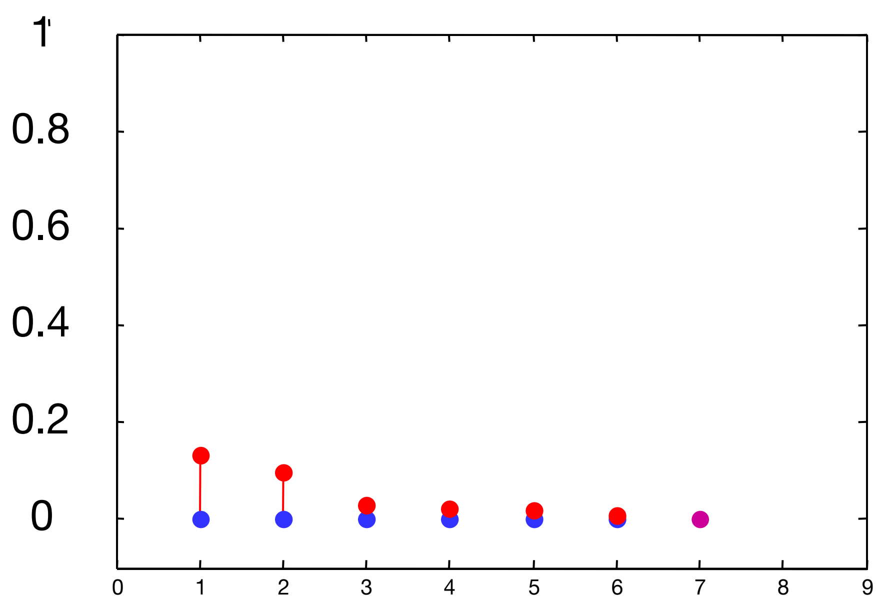
n=8

$$\sum a_i = 1$$

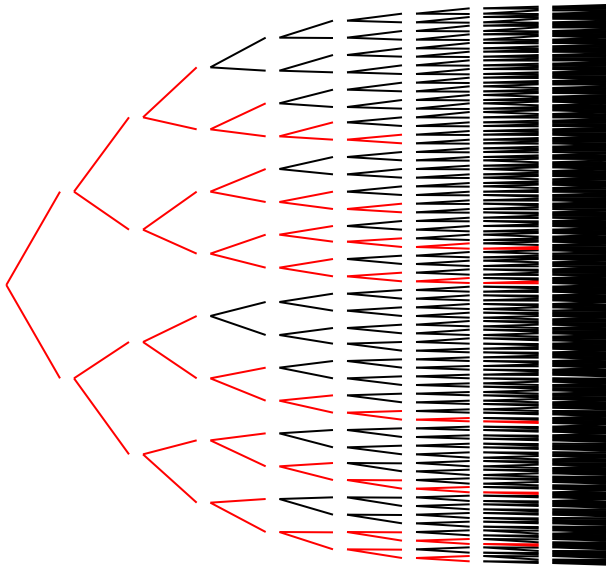
0.2116
0.1677
0.1358
0.1312
0.1307
0.1079
0.0892
0.0259



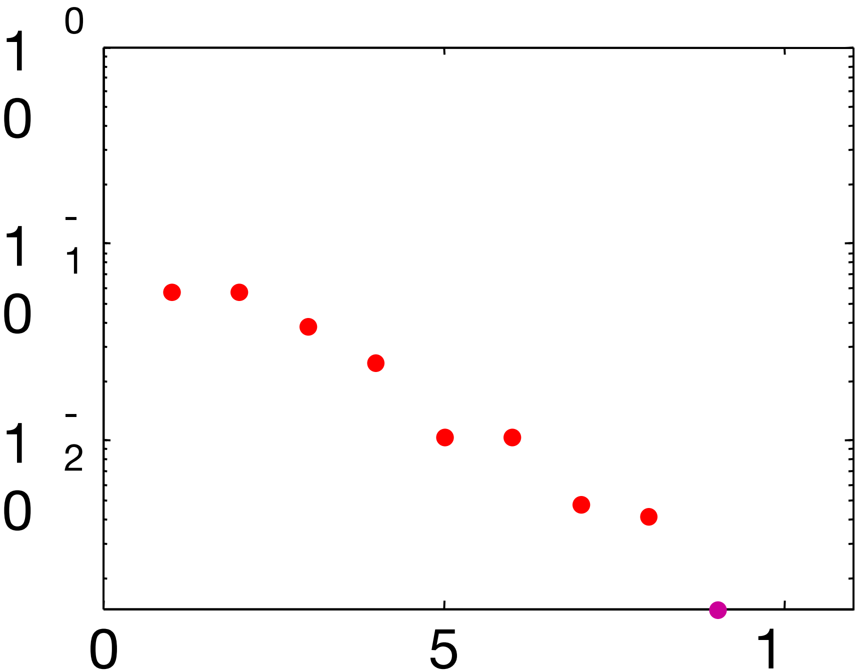




n=10

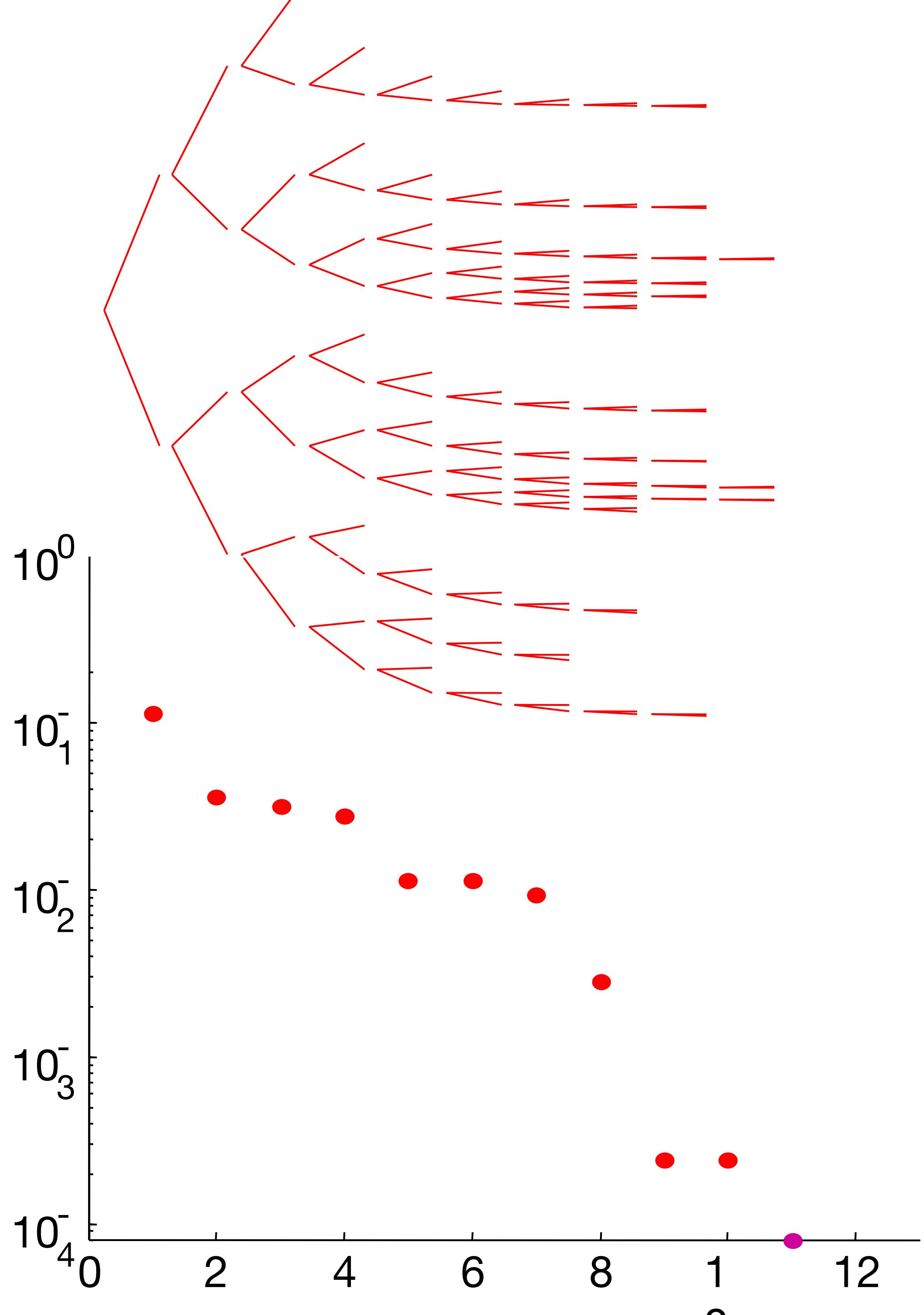


0.1552
0.1479
0.1448
0.1216
0.1204
0.1044
0.0932
0.0848
0.0149
0.0129

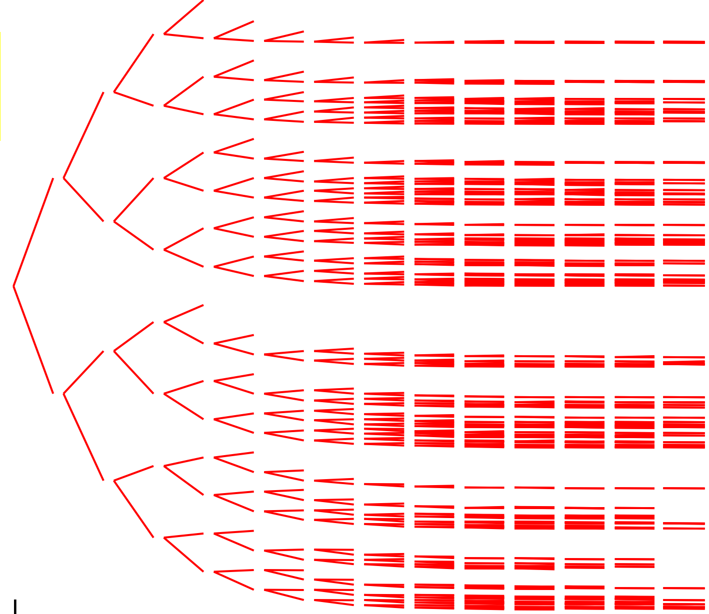


n=12

0.20900902553447
0.16372175132032
0.11474666241757
0.11060830317527
0.10264423321886
0.09262647734766
0.06575709562532
0.04944987218796
0.04533843900729
0.02356457821016
0.02025723346225
0.00227632849288

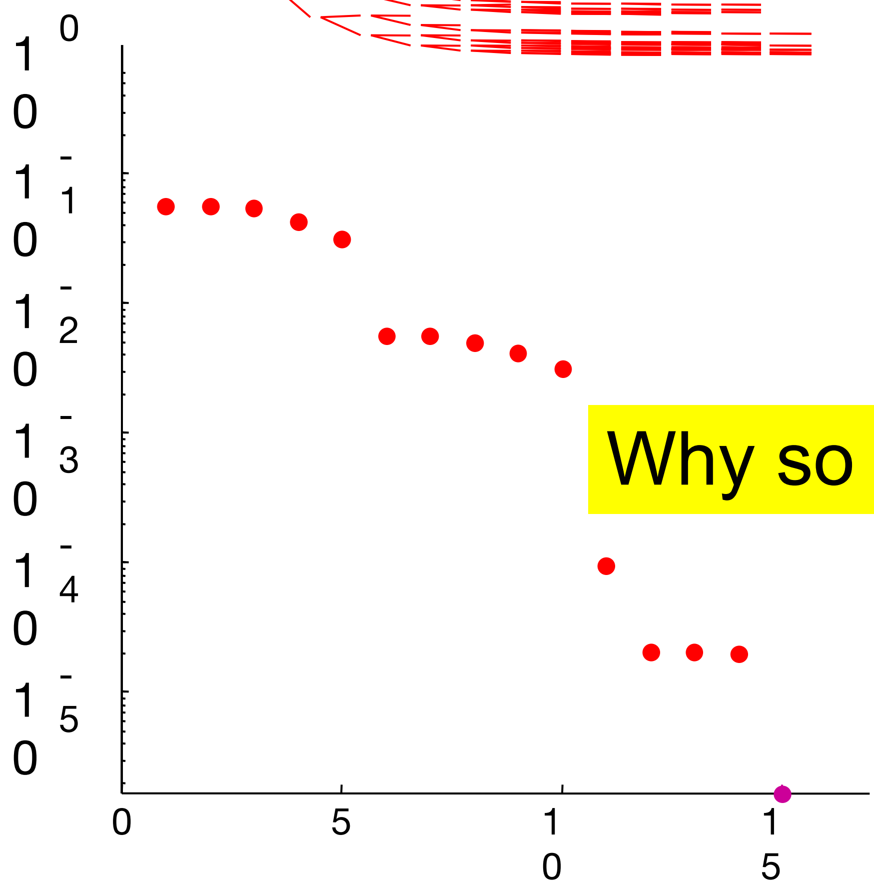


Still exponentially bad.



n=16

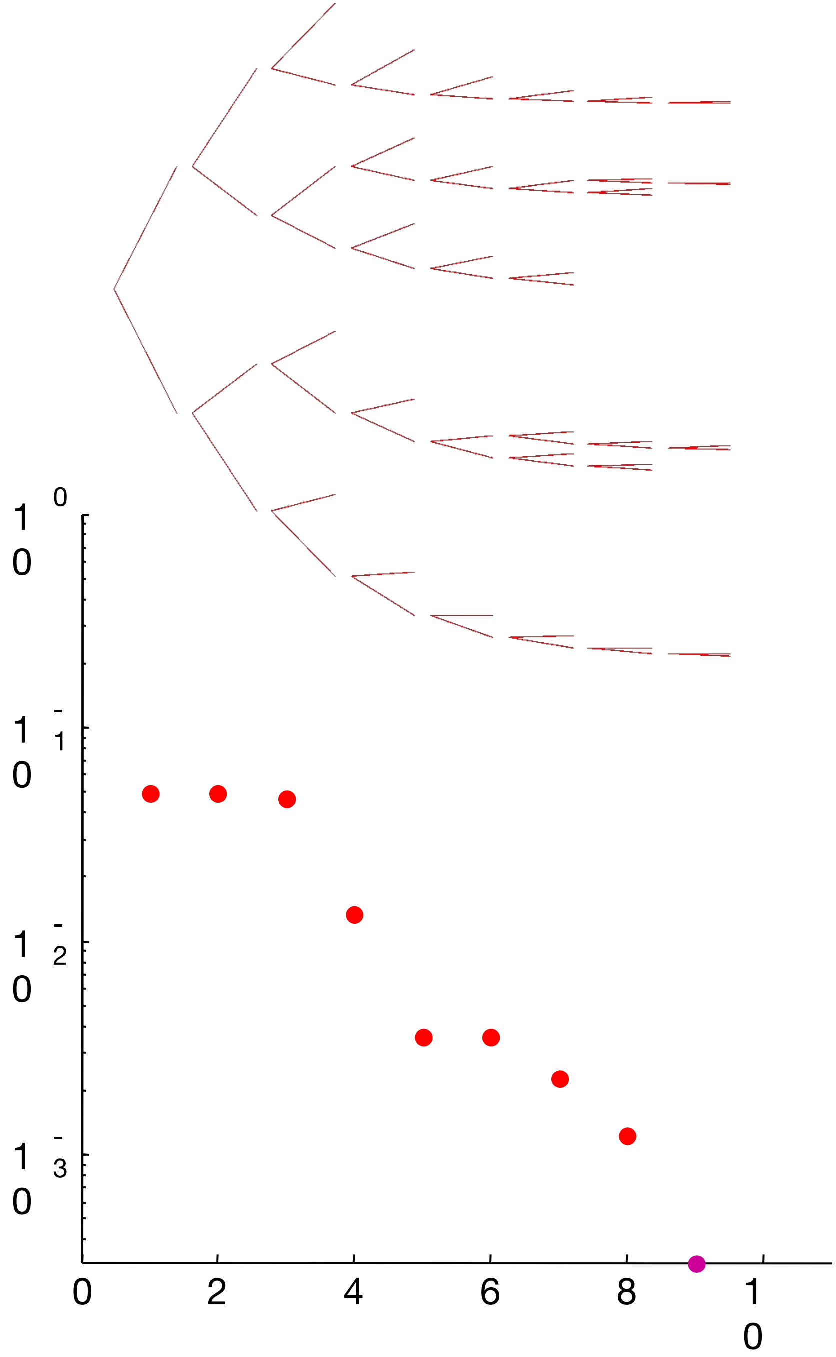
0.11874666798309
0.1121151292664‍7
0.11169327453340
0.11095064177068
0.09412186438521
0.08685317462754
0.08118017281551
0.06766995122518
0.05718523360114
0.03754549903682
0.03586488042322
0.03254947795691
0.01521112174069
0.01506074475625
0.01423812298937
0.00901404288853



Why so hard?

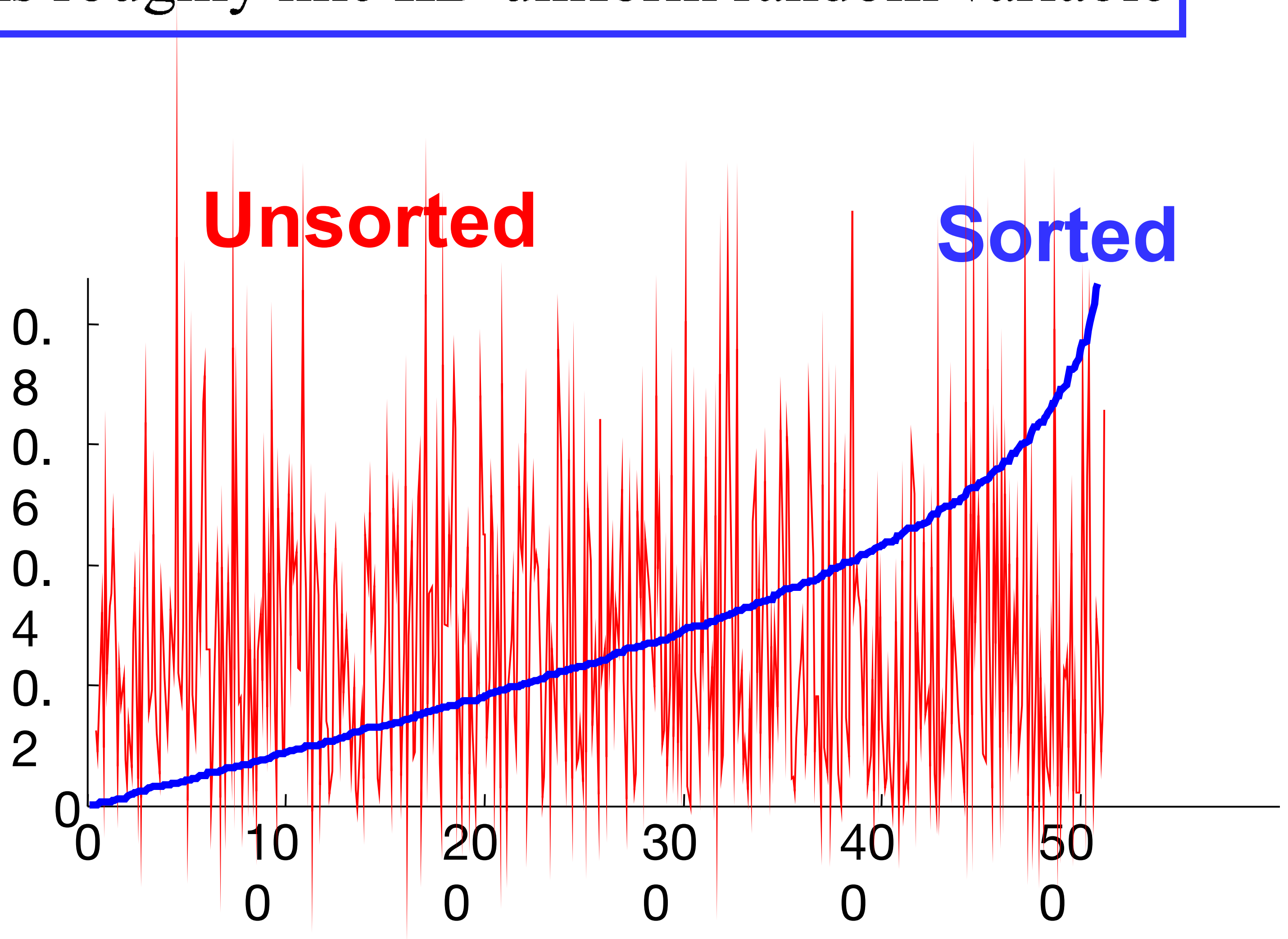
n=10

0.16212567898594
0.14166406741328
0.13672813657519
0.13542304261100
0.11591869442981
0.11370146803691
0.06062893005904
0.05985800729769
0.04919688814988
0.02475508644126



$\left|\sum a_i x_i\right|$ for $x_i = \pm 1$

looks roughly like IID uniform random variable



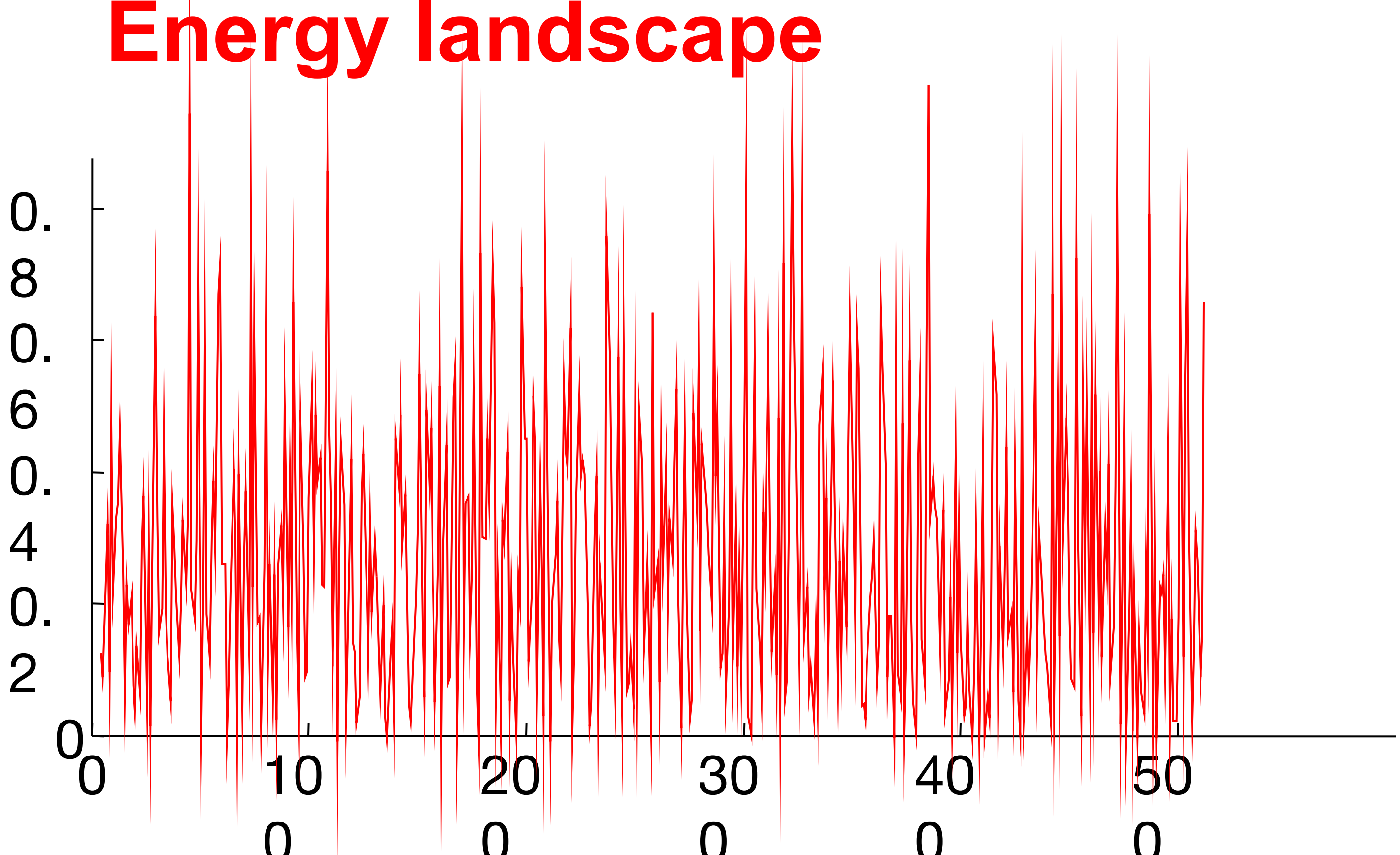
n=10

$$\left|\sum a_i x_i\right|^2 = \sum \left(a_i a_j\right) x_i x_j$$

antiferromagnetic Mattis spin glass

$\left(\min\right)^2$ =ground state energy

Energy landscape



$$E\left(\min \text{of } m \text{ IID uniform } [0,1] \text{ random variables}\right) \propto \frac{1}{m}$$

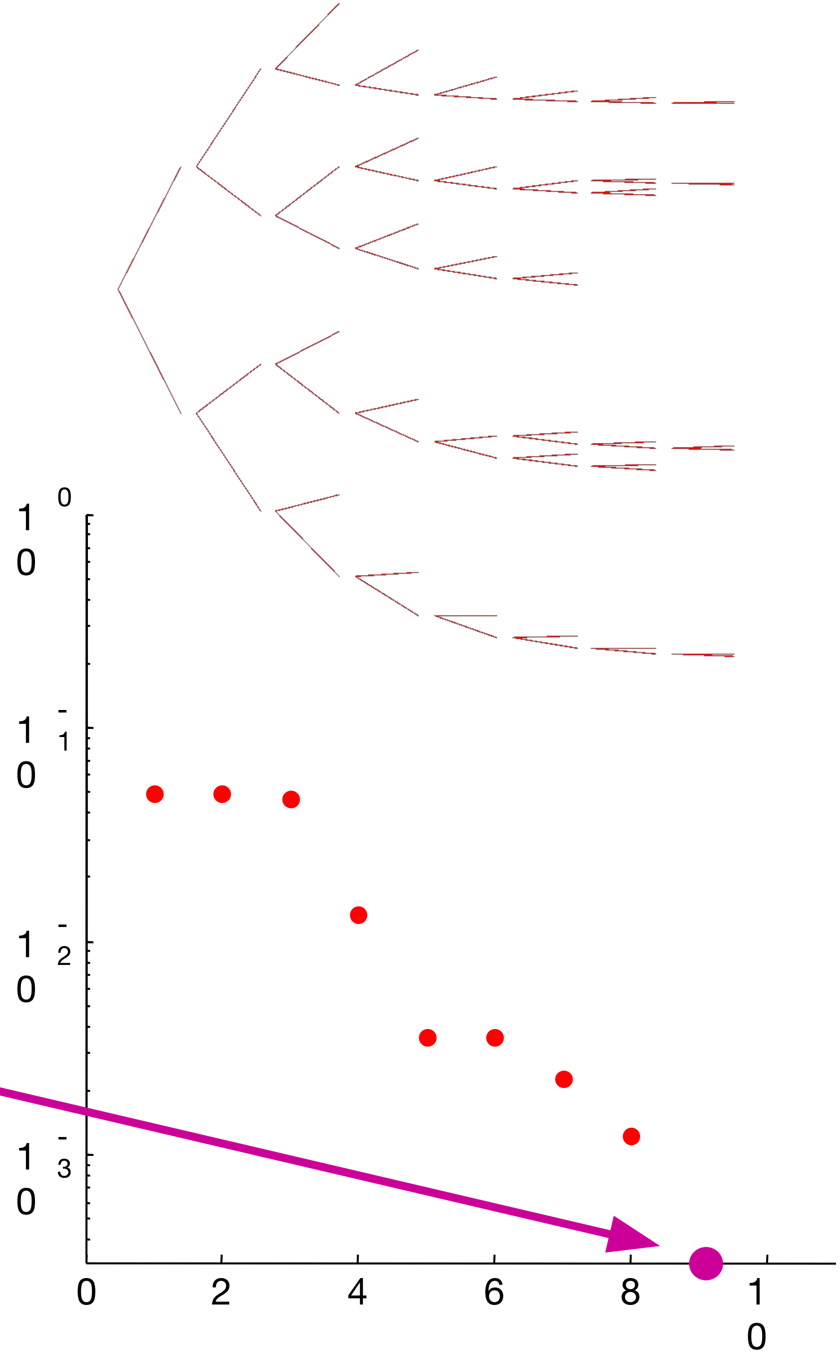
$$E(\min \text{of } 2^m \text{ IID uniform } [0,1] \text{ random variables}) \propto 2^{-m}$$

$$E\left(\min_{x_i^2=1} \frac{\left|\sum a_i x_i\right|}{\sum a_i}\right) \propto \frac{2^{-n}}{\sqrt{n}}$$

Large statistical physics literature

Mertens, Derrida, Gross & Mezard, …

n=10

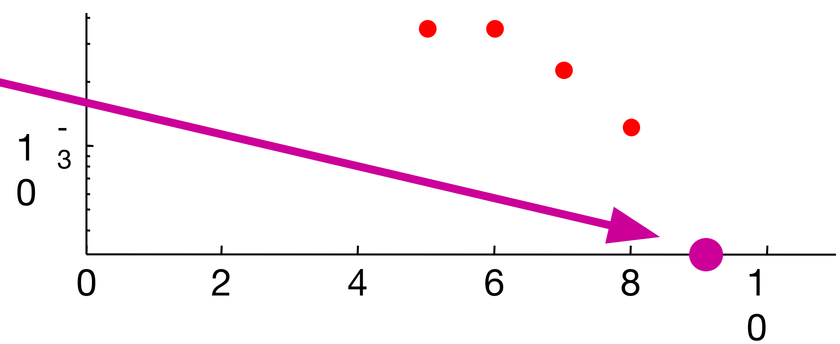


$$E\left(\min_{x_i^2=1}\frac{\left|\sum a_i x_i\right|}{\sum a_i}\right)\propto\frac{2^{-n}}{\sqrt{n}}$$

$$\min_{x_i^2=1} \frac{\left|\sum a_i x_i\right|}{\sum a_i} > \varepsilon? \qquad \text{where} \qquad \varepsilon << \frac{2^{-n}}{\sqrt{n}}$$

This is true "almost surely," but there is currently no method that will systematically generate short proofs.

$$E\left(\min_{x_i^2=1} \frac{\left|\sum a_i x_i\right|}{\sum a_i} \right) \propto \frac{2^{-n}}{\sqrt{n}}$$



Why should anyone care?

- Computational problems in biology and advanced technologies are even harder.
- If we can't do this "simple" problem, what hope is there for scalability of computational methods to large networks?
- Is there some other reason for optimism?

Toy problem: $$R = \min_{x_i^2=1} \frac{\left|\sum a_i x_i\right|}{\sum a_i}$$

In general: $$R = \min_{x \in X} M(x)$$

M = model of system

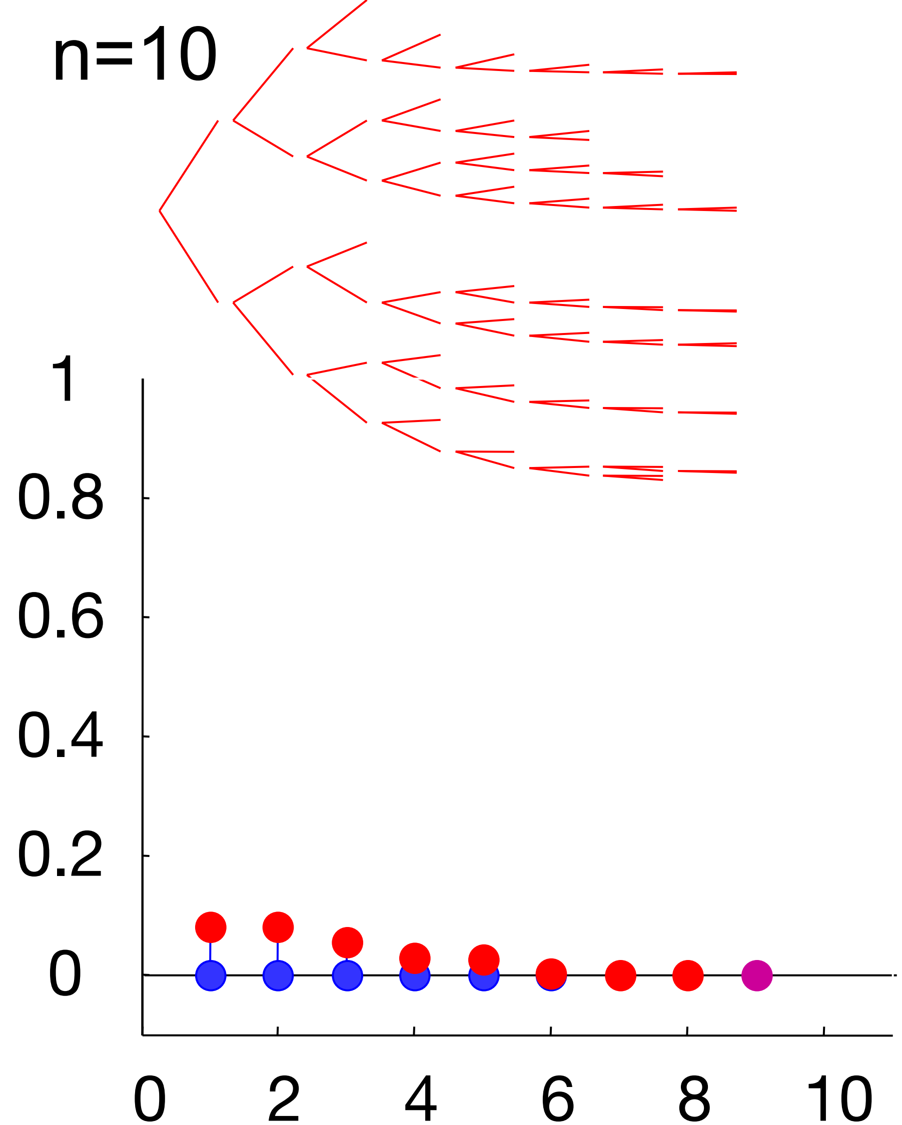
X = uncertainty set

R = robustness

Central computational problems in biology and advanced technologies can be written this way and are formally "hard" (NP/coNP hard or undecidable)

Robustness

$$R = \min_{x_i^2=1} \frac{\left|\sum a_i x_i\right|}{\sum a_i}$$



C = search depth

Complexity

$$\sum a_i = 1$$

Various levels of paranoia

$$R = \min_{x_i^2=1} \left|\sum a_i x_i\right|$$

explicitly modeled uncertainty

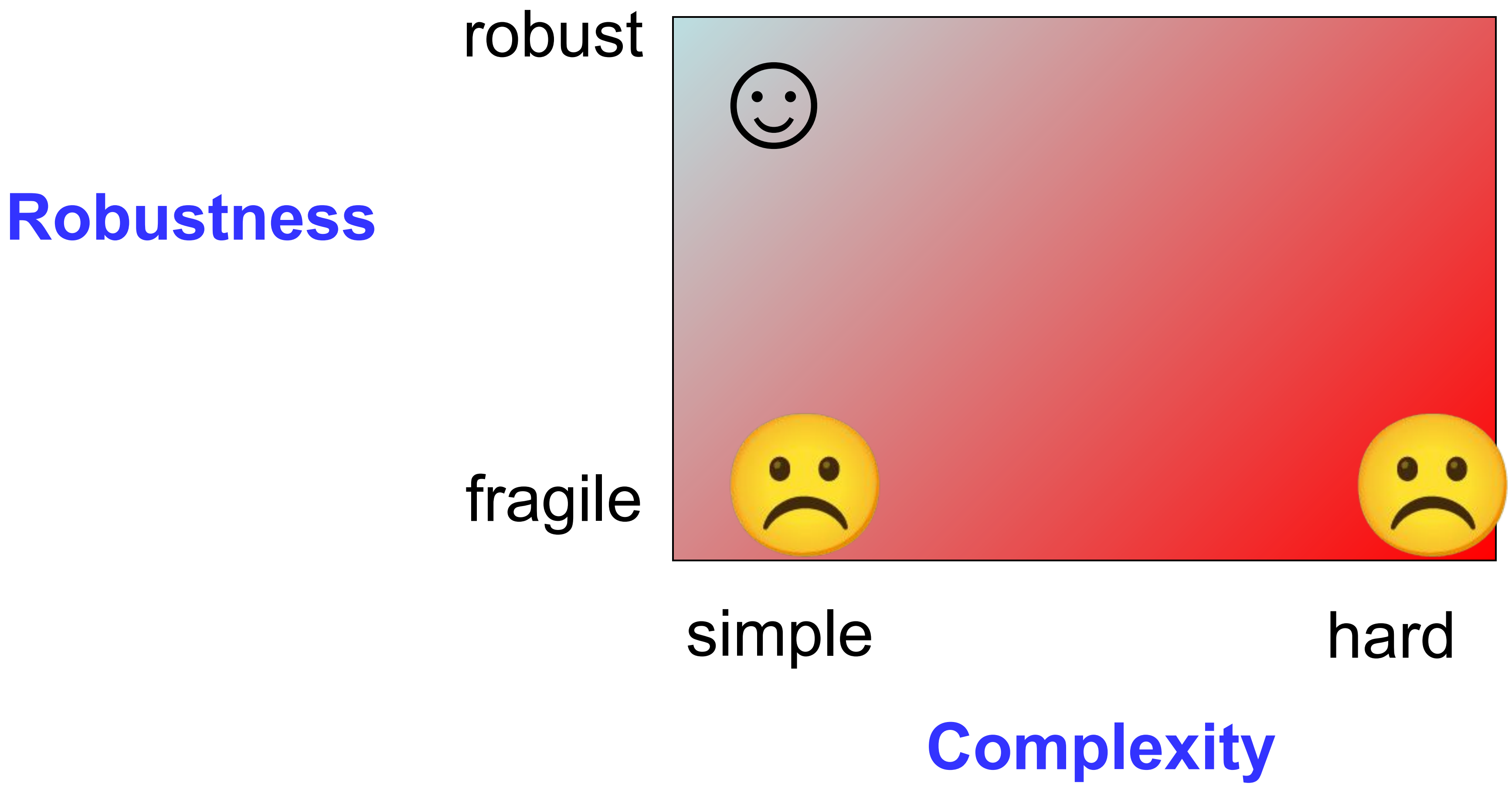
$$R_2 = \left\{ \min \sum |a_i - b_i| \;\middle|\; \min_{x_i^2=1} \left|\sum b_i x_i\right| = 0 \right\}$$

$$R_3 = \left\{ \min \delta \;\middle|\; \min_{\left||x_i|-1\right| \le \delta} \left|\sum a_i x_i\right| = 0 \right\}$$

check for fragilities to model parameters

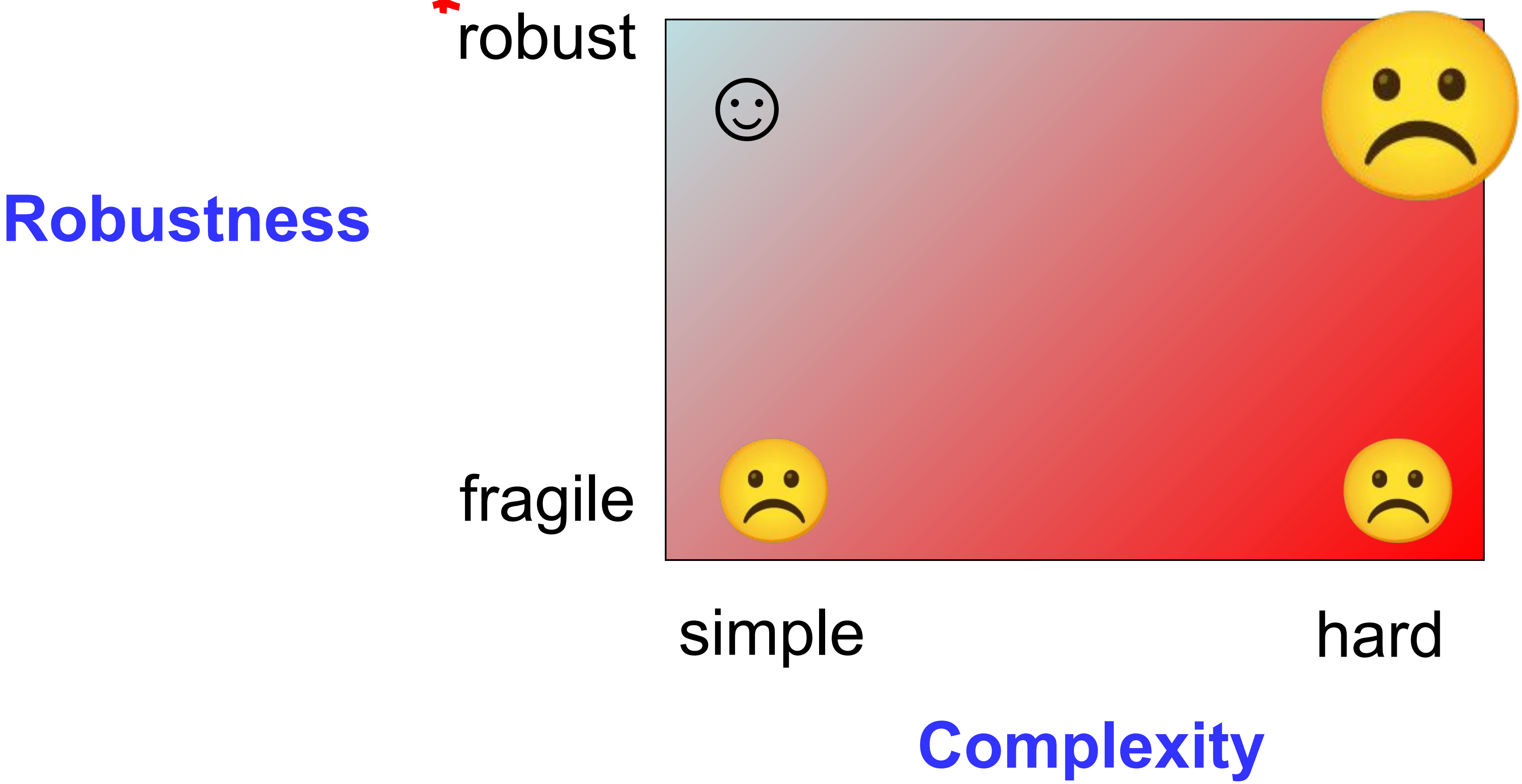
$$\boxed{\text{Theorem: } R = R_2 = R_3}$$

In general



NIGHT MARE?

What if robust systems are intrinsically hard to verify and understand?



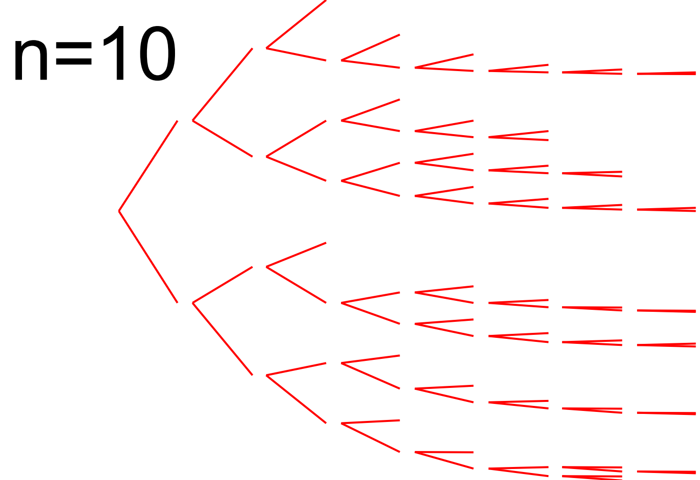
NIGHT MARE?

What if robust systems are intrinsically hard to verify and understand?

Biology: We might accumulate more complete parts lists but never "understand" how it all works.
Technology: We might build increasingly complex and incomprehensible systems which will eventually fail completely yet cryptically.

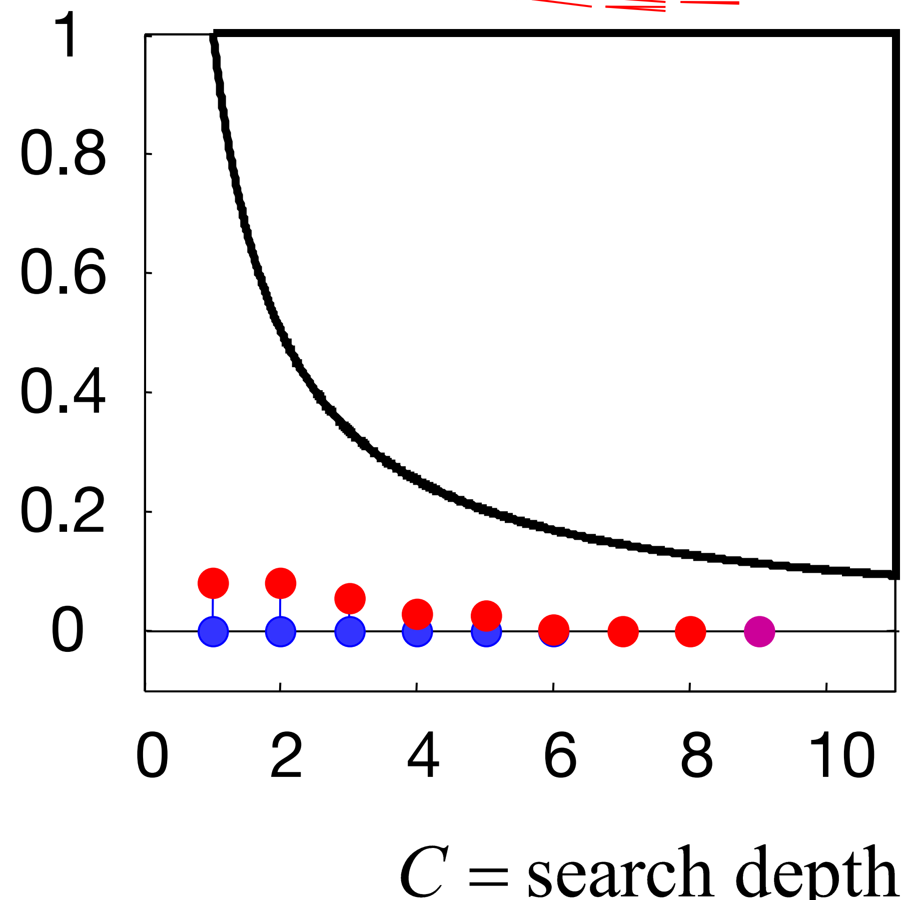
- Nothing in the orthodox views of complexity says this won't happen (apparently).
- Fortunately, there is some good news.
- Illustrate the "good news" in our simple problem.

Theorem: $C \leq \dfrac{1}{R}$



$$R = \min_{x_i^2=1} \frac{\left|\sum a_i x_i\right|}{\sum a_i}$$

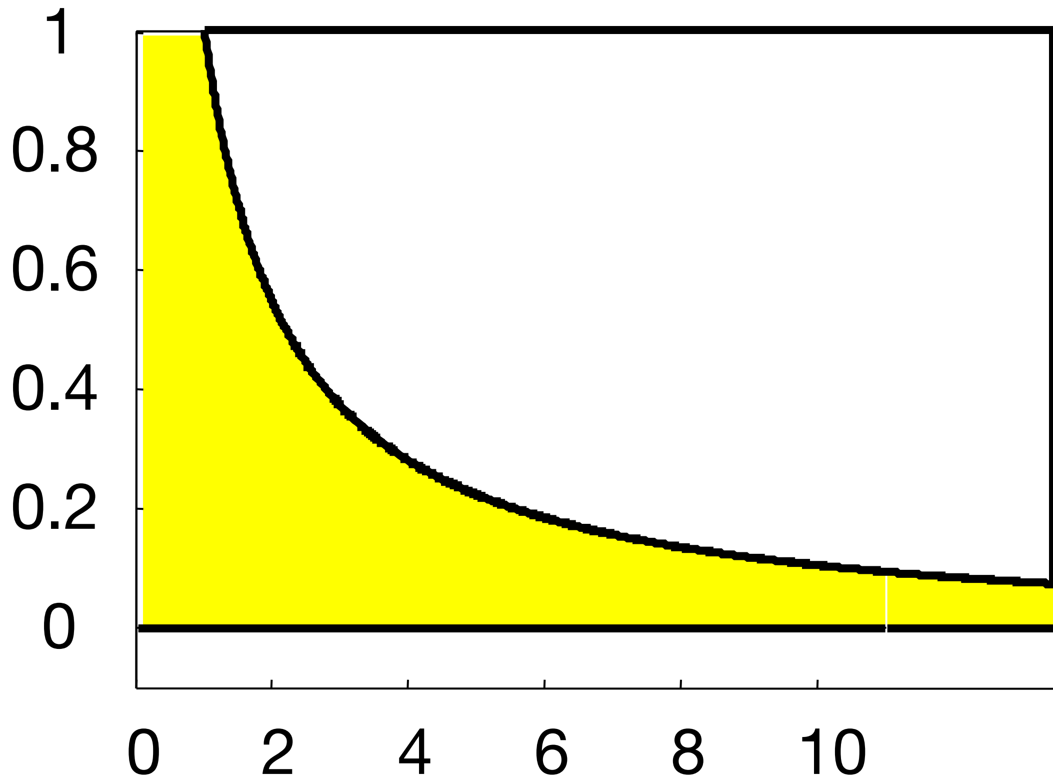
$$C \leq \frac{1}{R} = "Fragility"$$



Theorem: $C \le \dfrac{1}{R}$

There exist example instances in all the regions permitted by the theorem.

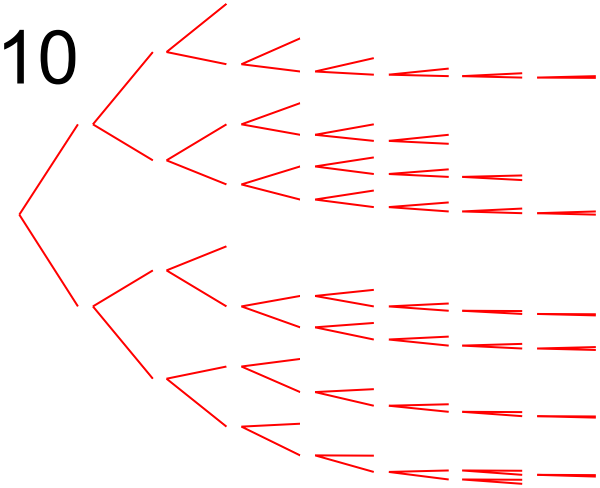
$$C \le \frac{1}{R} = "Fragility"$$

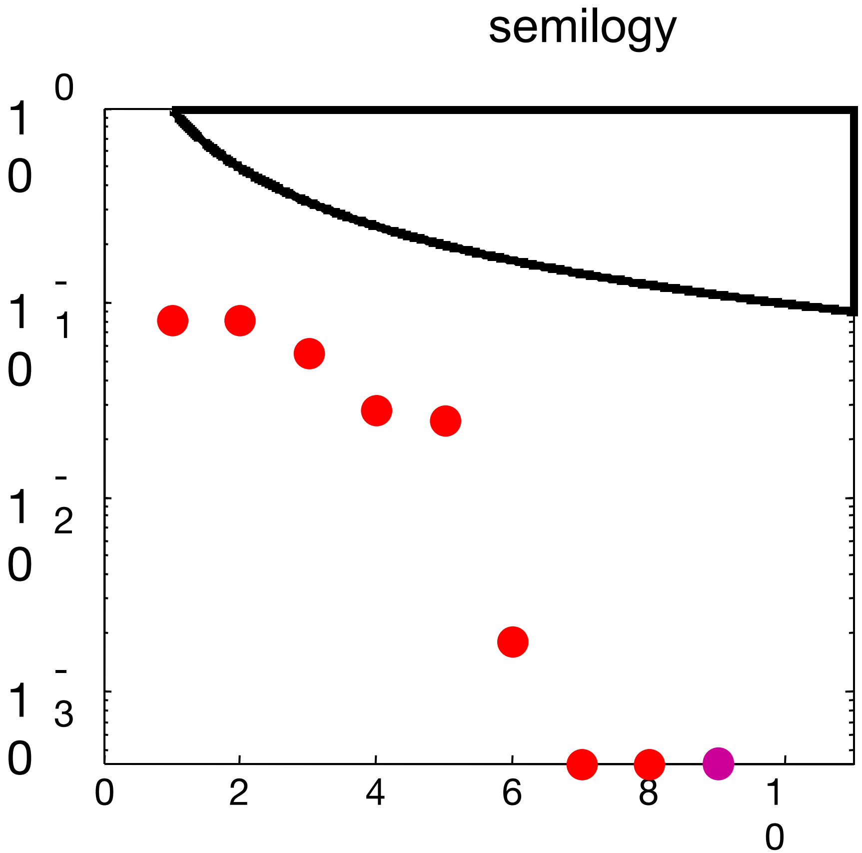


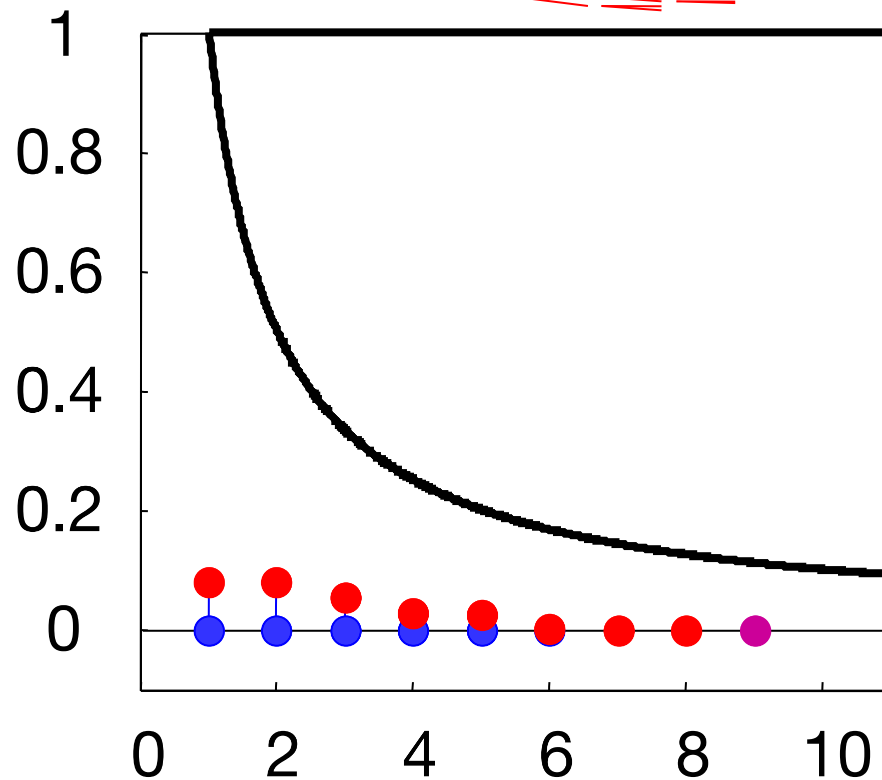
C = search depth

Theorem: $C \leq \frac{1}{R}$

n=10

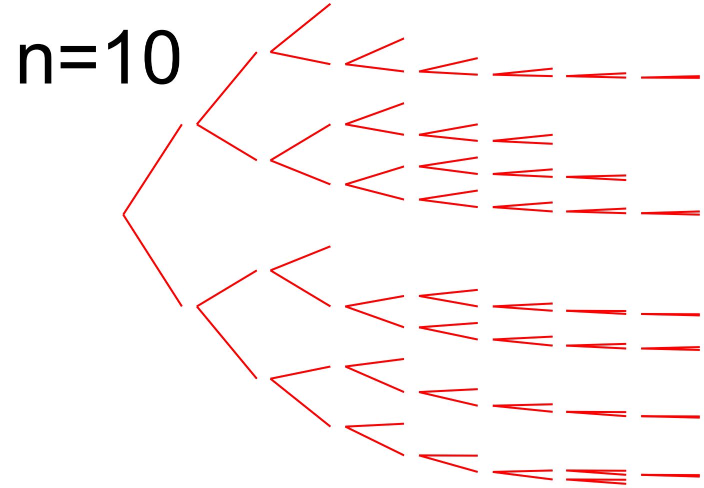


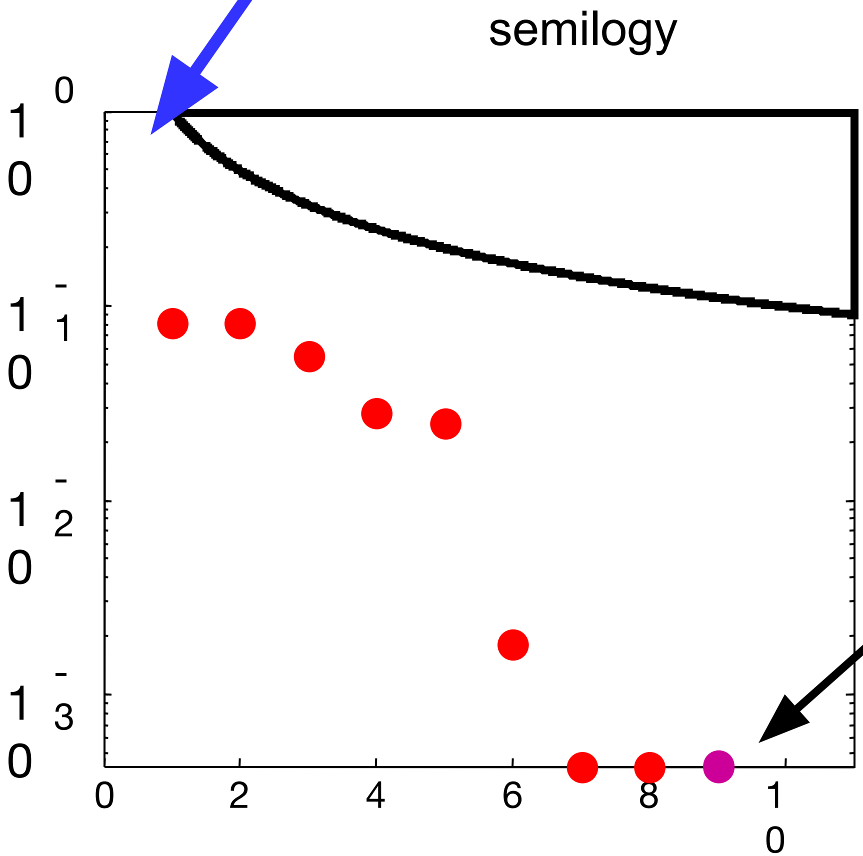




C = search depth

Robust problems are rare and highly structured

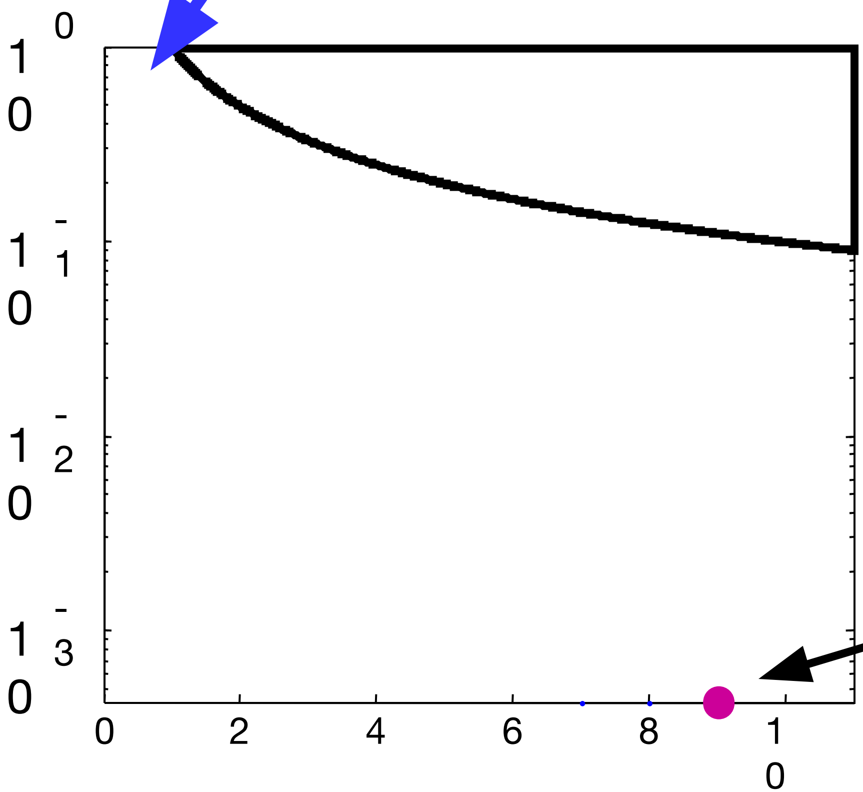




Random problems are highly complex and extremely fragile

Computing is HARD at phase boundaries.

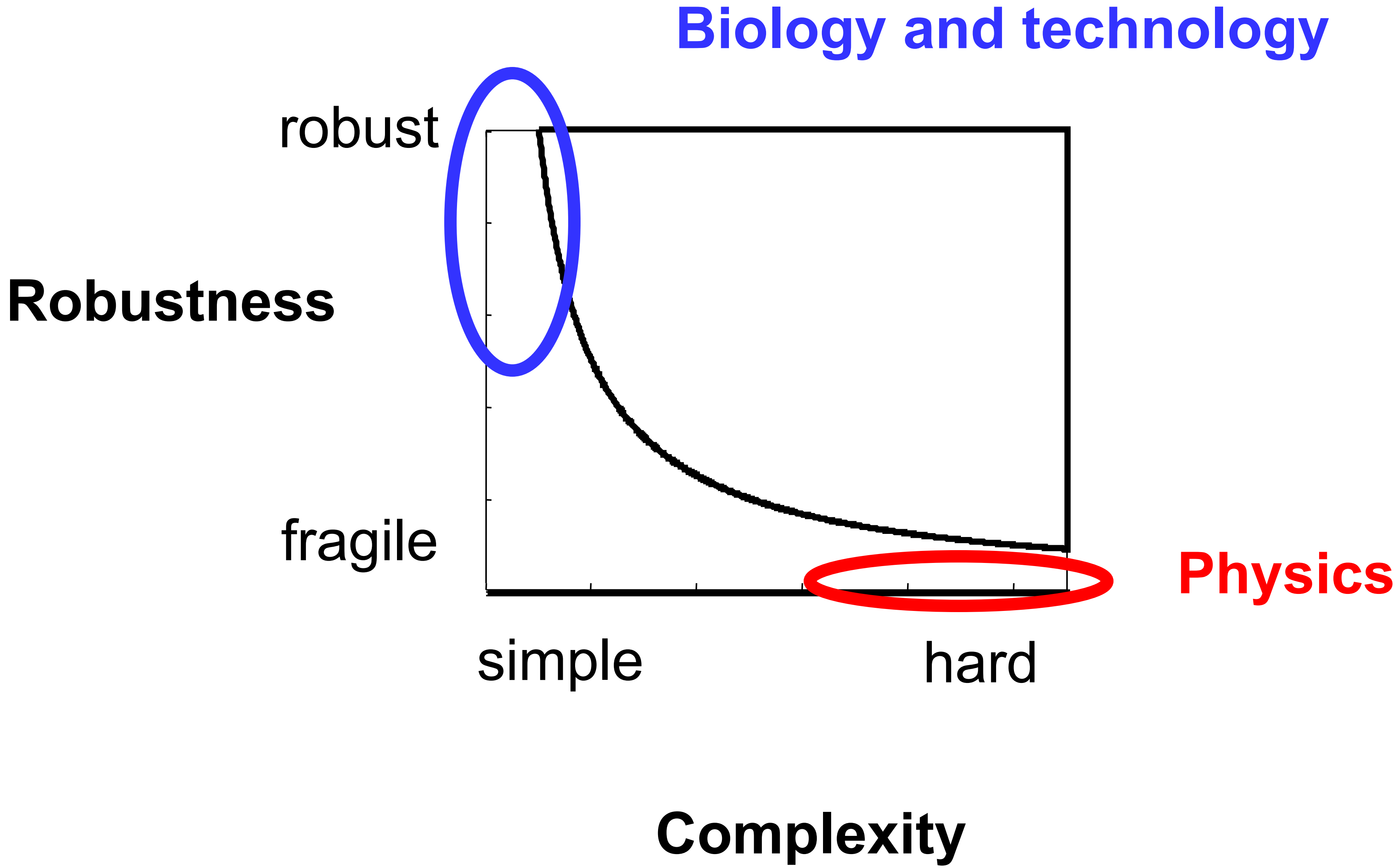
Robust problems are rare and highly structured



The most "interesting" problems are on the boundary.

Random problems are highly complex and extremely fragile.

Theorem: $C \leq \dfrac{1}{R}$



Theorem: $C \le \frac{1}{R}$

$$\left.\begin{array}{l} C = \text{ search depth} \\ \# = \text{operation count} \end{array}\right\} \Rightarrow \quad \# \le n\,2^{C}$$

$$C \le \frac{1}{R} \quad \Rightarrow \quad \boxed{\# \le O\left(n\,2^{\frac{1}{R}} \right)}$$

$$\begin{array}{c} \text{Linear} \\ \text{Program} \end{array} \Rightarrow \quad \# \le O\left(n^2 \log\left(\frac{1}{R} \right) \right)$$

Random problems

$$\min_{x_i^2=1}\left|\sum a_i x_i\right| \approx \frac{2^{-n}}{\sqrt{n}}$$

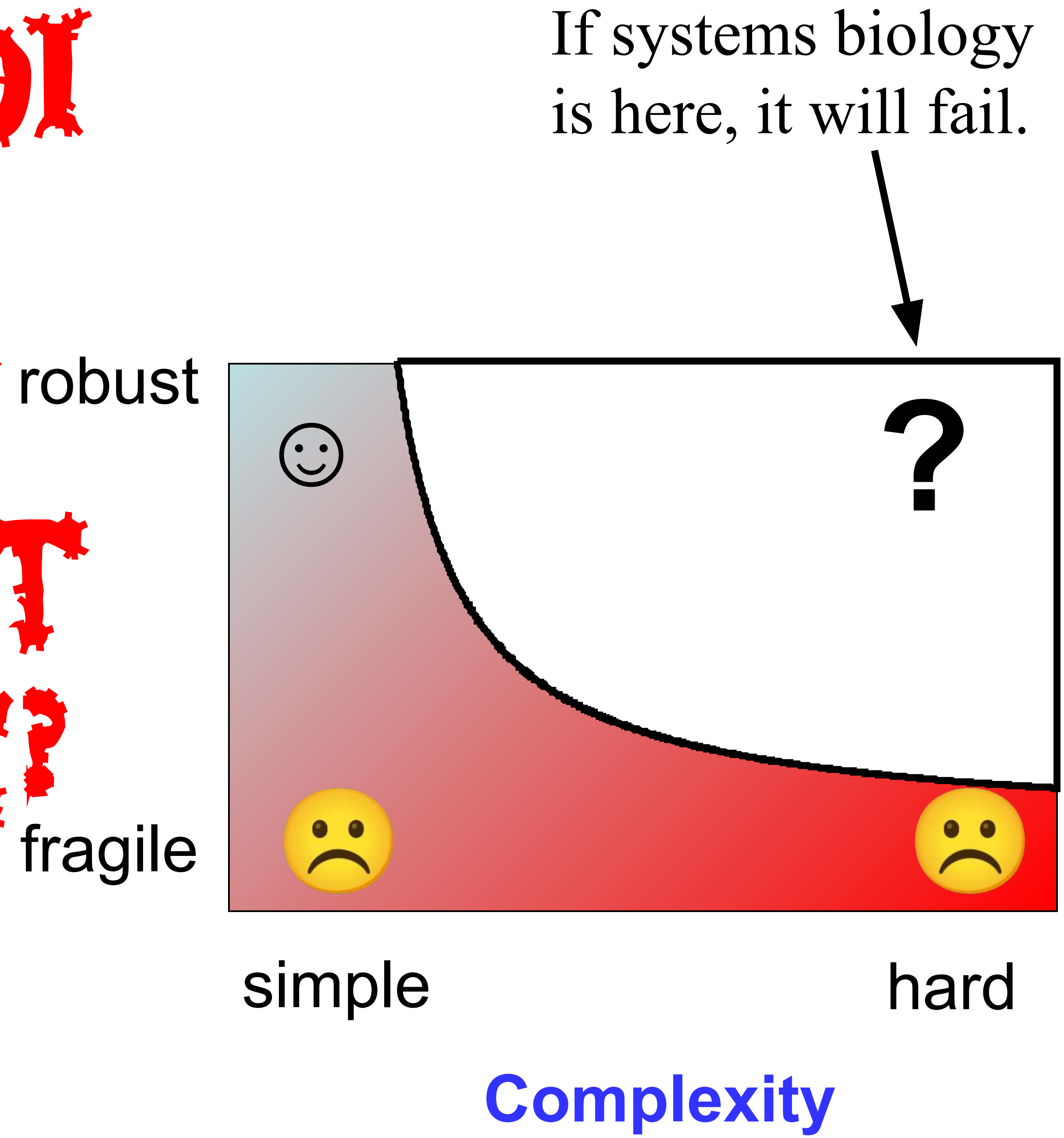
This is true "almost surely," but has proof length of $\# = O\left(2^n\right)$

Random problems are hopelessly fragile, and it's easy to show that:

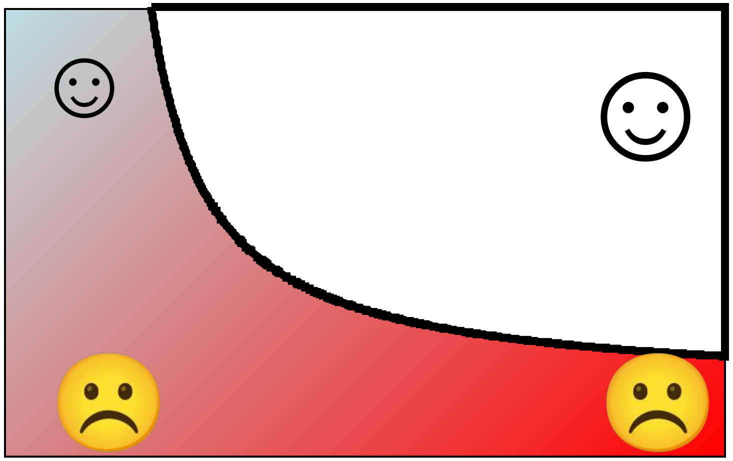
Robust problems

$$\# \le O\left(n\, 2^{\frac{1}{R}}\right)$$

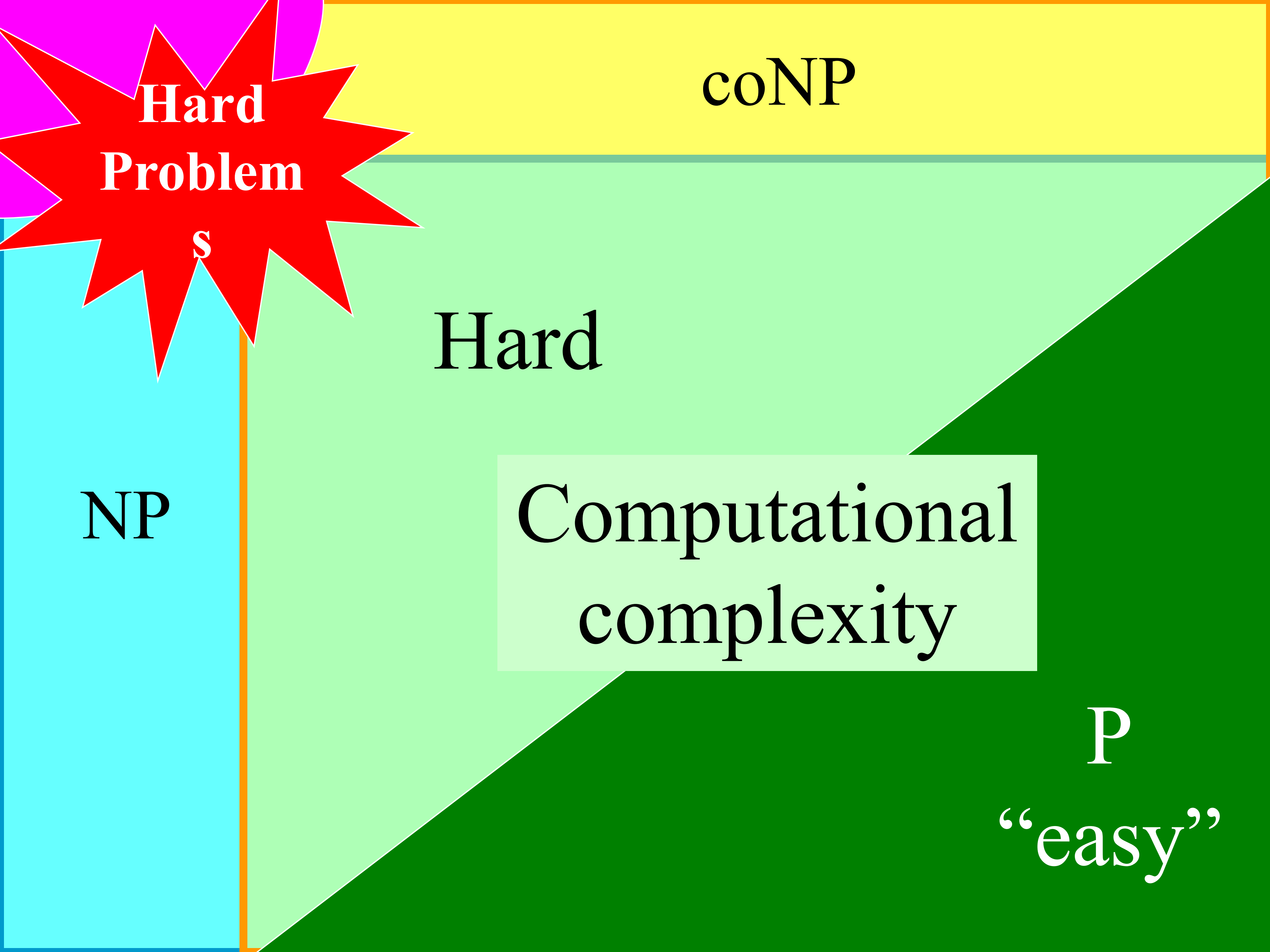
AVOIDING THE NIGHTMARE?



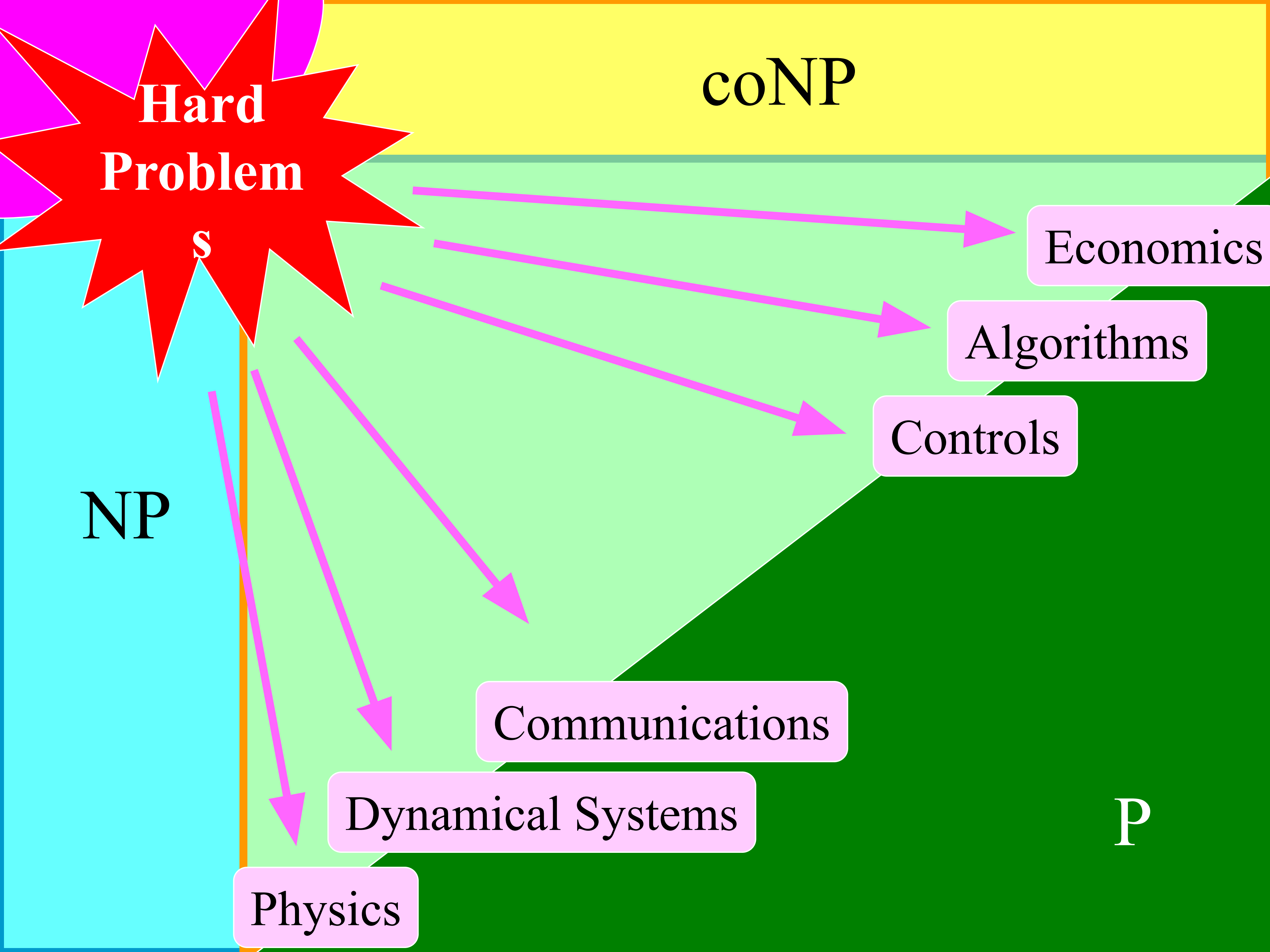
AVOIDING THE NIGHTMARE?



- Must decouple organizational and computational complexity
- Must explicitly exploit robustness/fragility in computation
- Substantial recent progress
- Small tip of a huge and growing iceberg, yet…
- New approach in its infancy



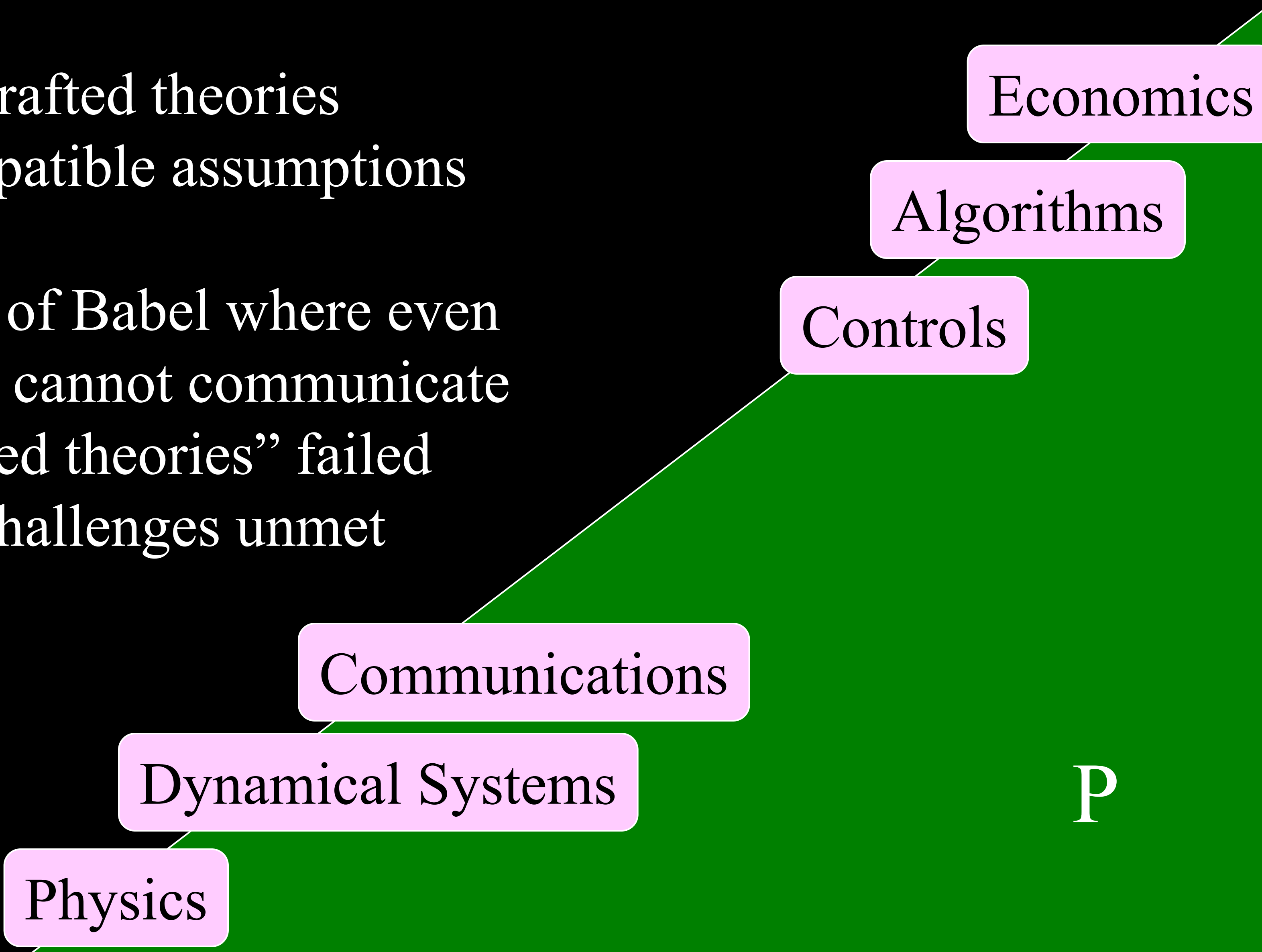
coNP
Hard Problems
Hard
NP
Computational complexity
P
“easy”

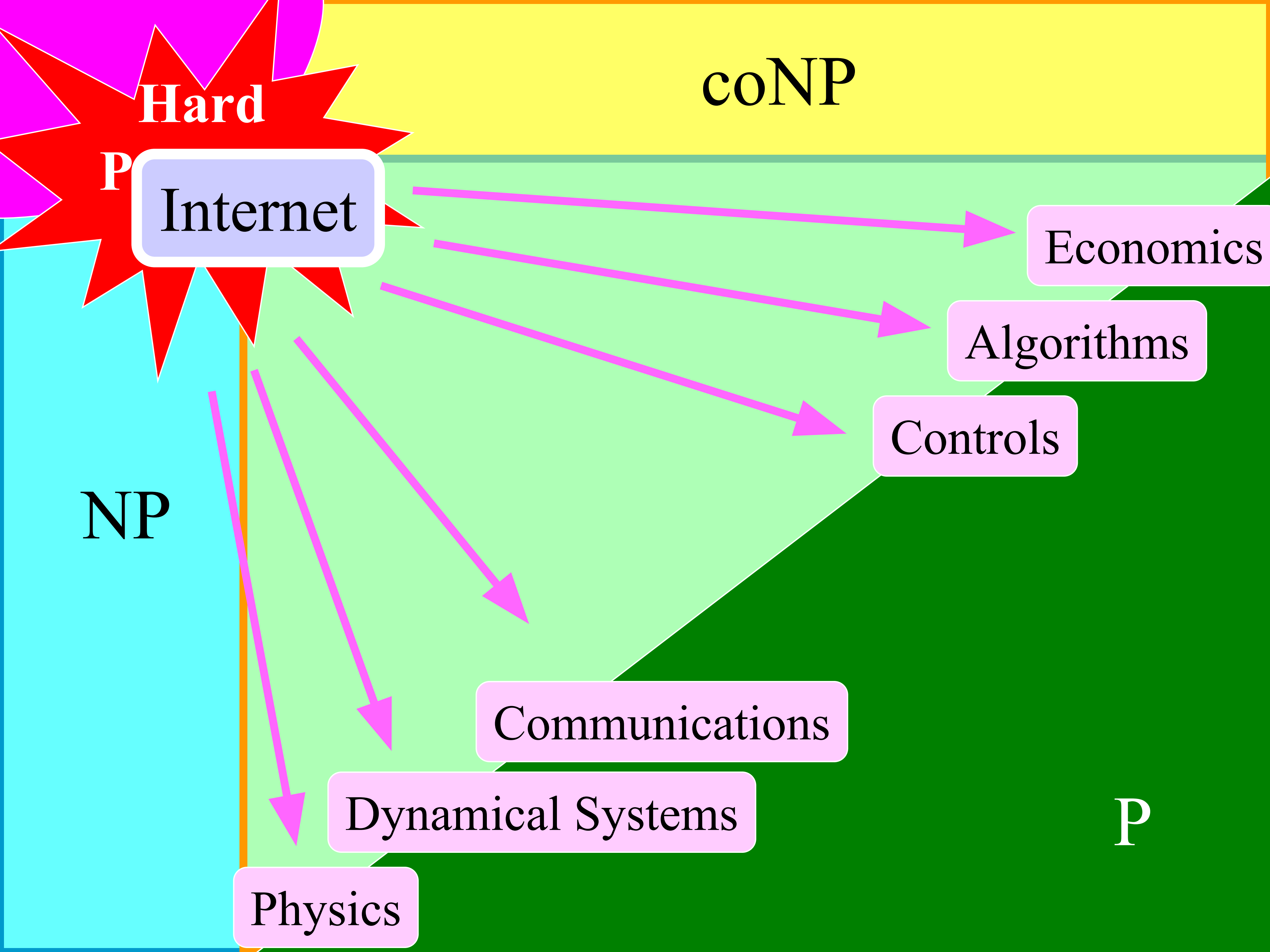

coNP
Hard
Problem
s
Economics
Algorithms
Controls
NP
Communications
Dynamical Systems
Physics
P

- Domain-specific assumptions
- Enormously successful

- Handcrafted theories
- Incompatible assumptions

- Tower of Babel where even experts cannot communicate
- “Unified theories” failed
- New challenges unmet

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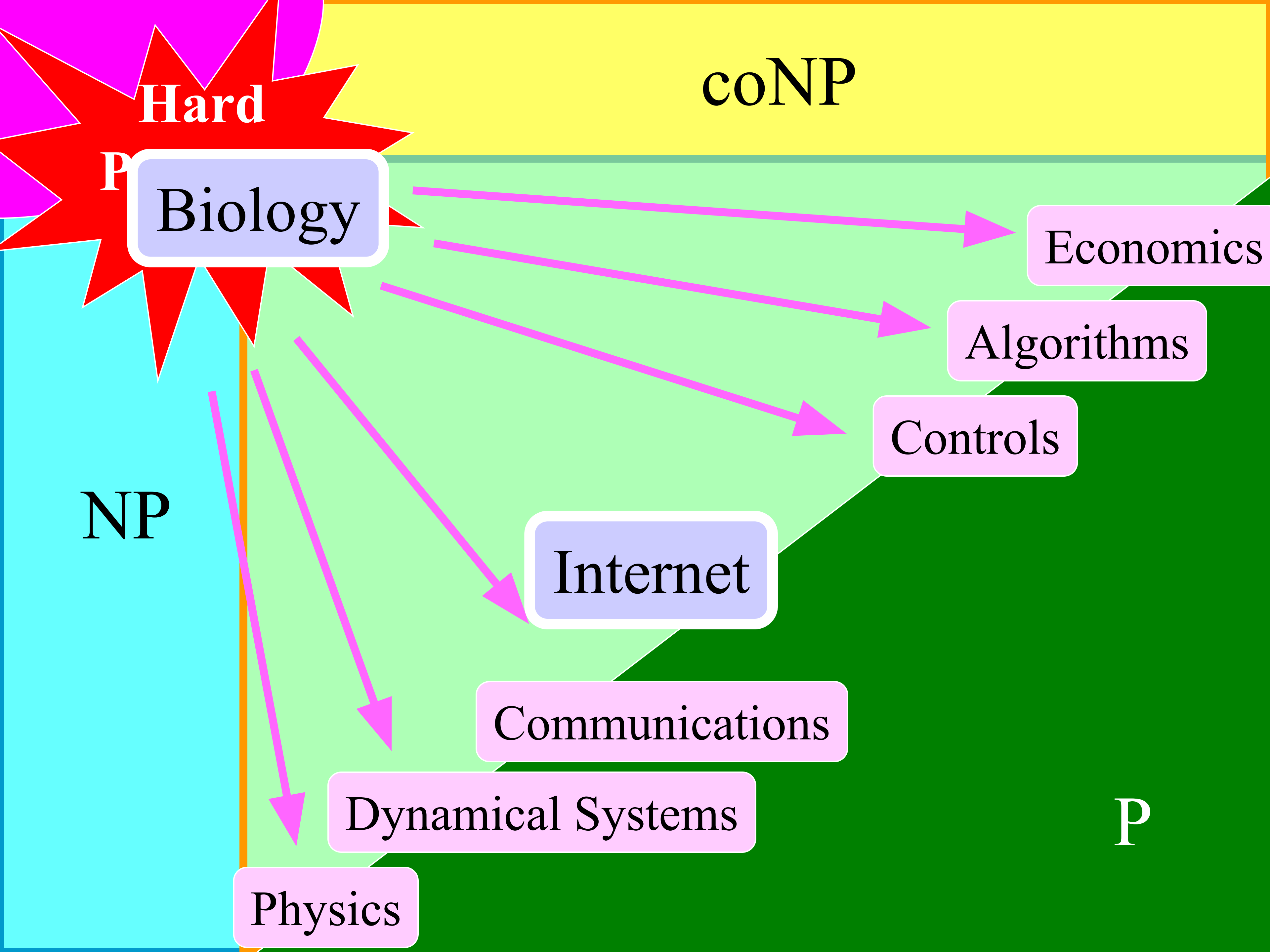
Physics
P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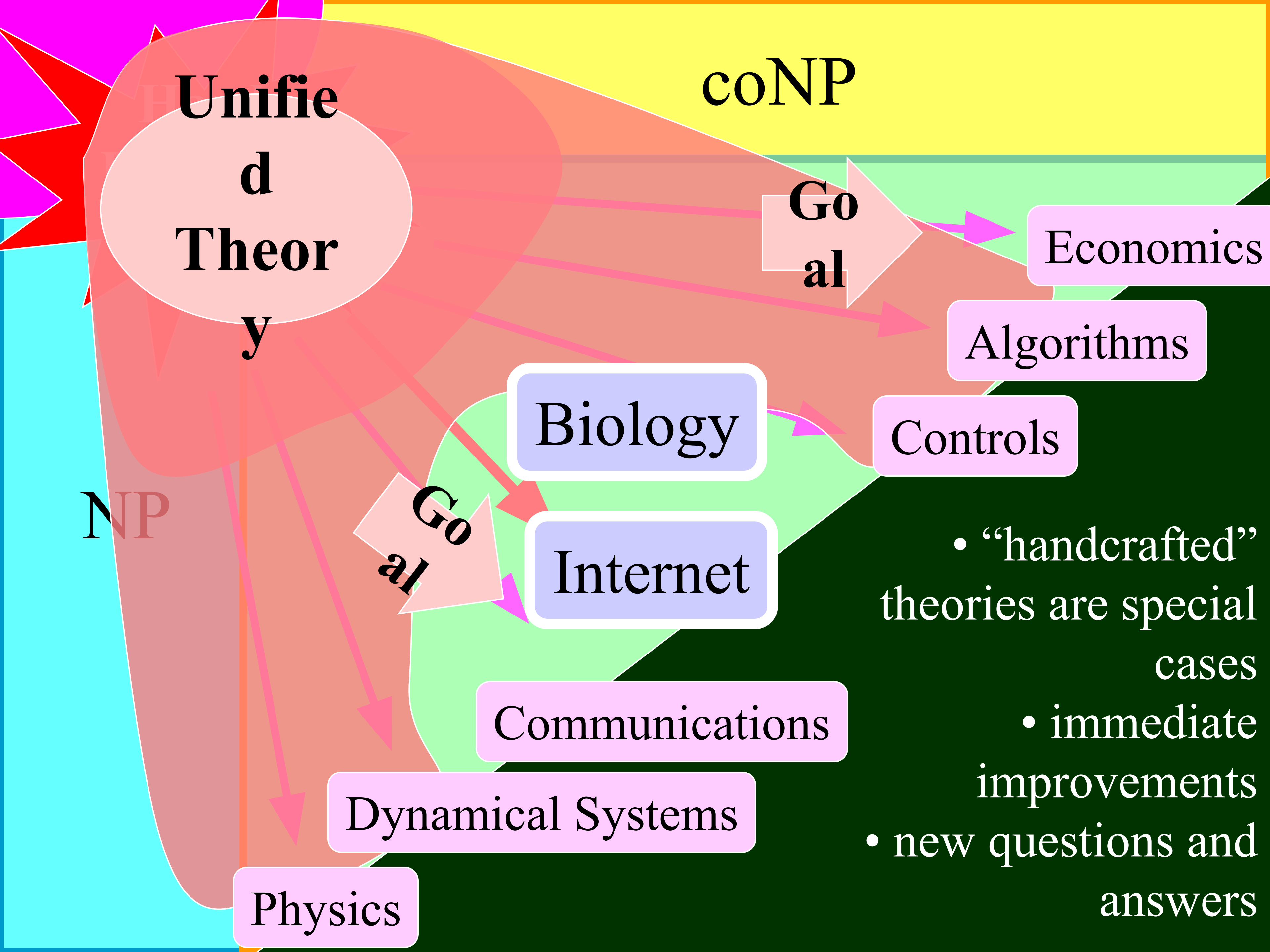
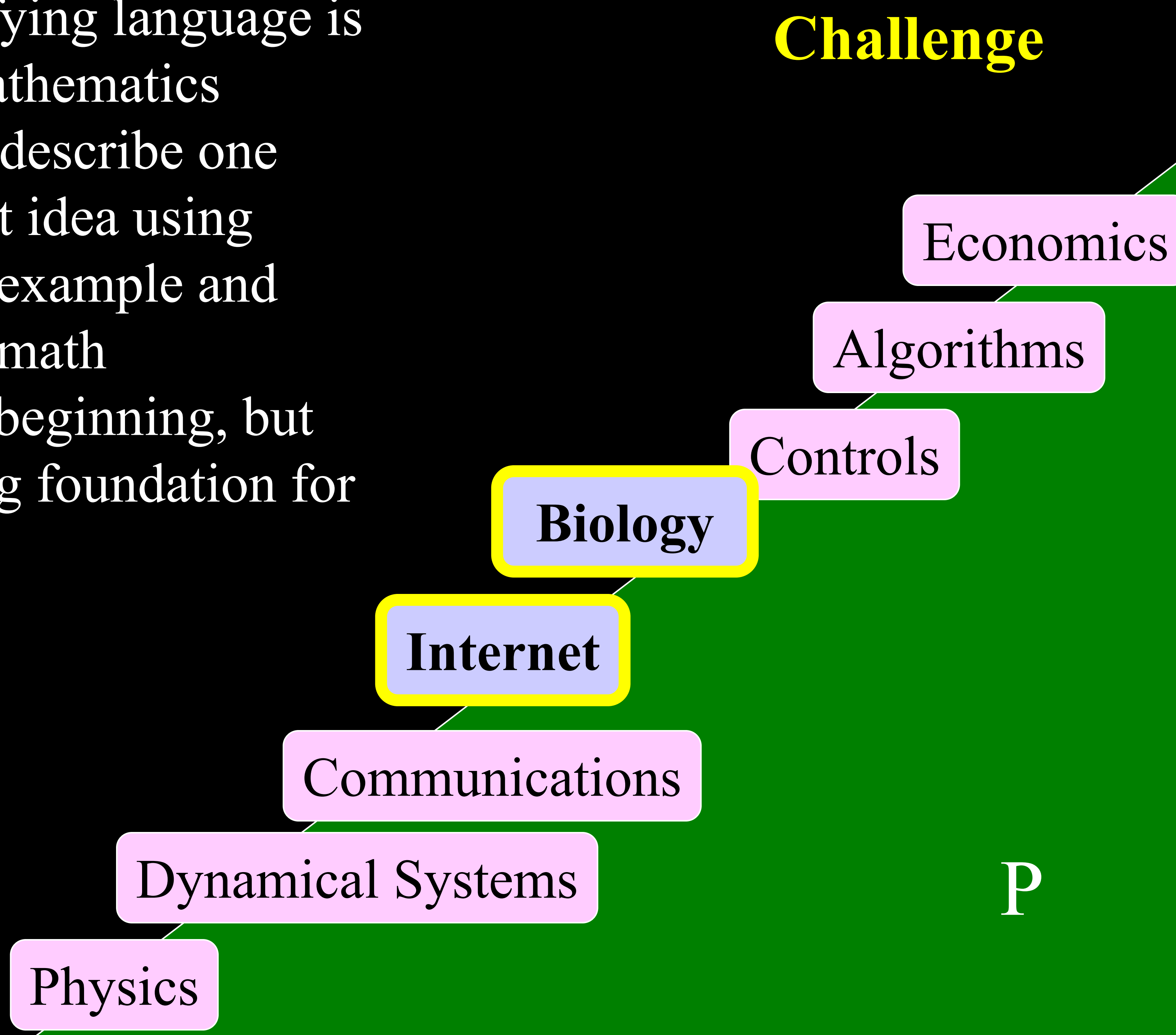
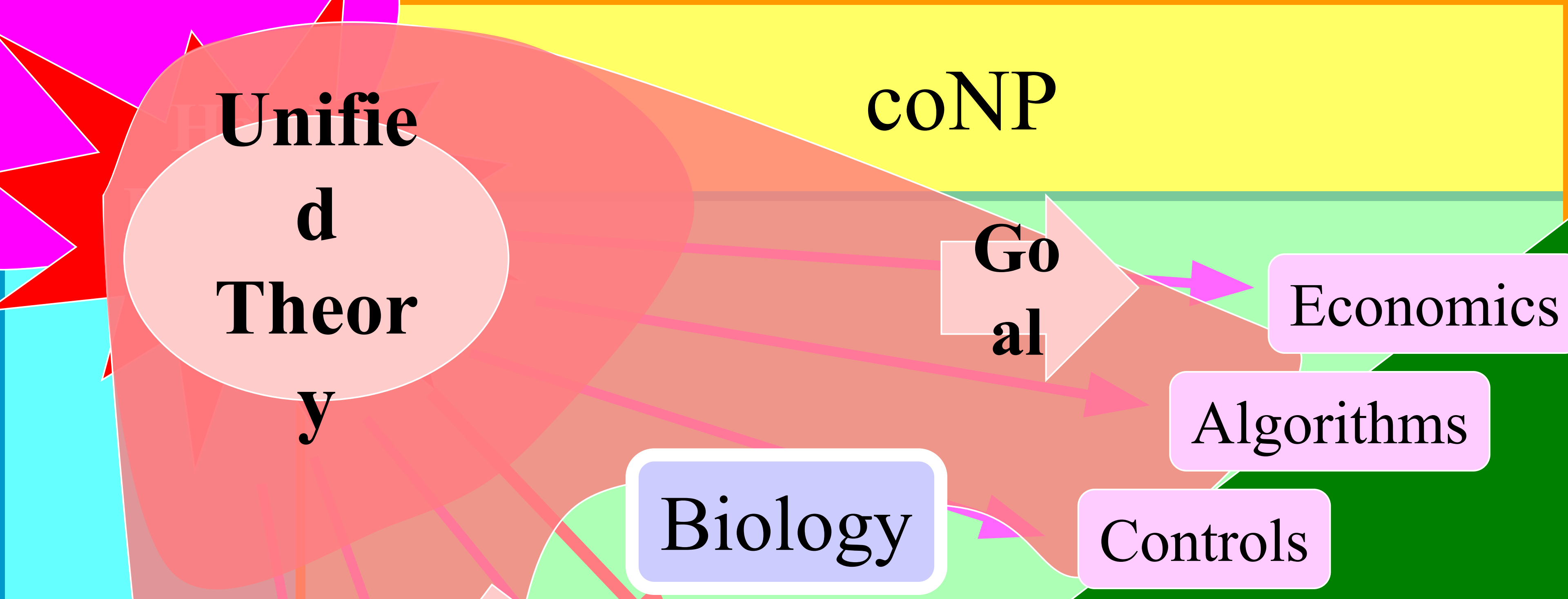


coNP
Unified Theory
Goal
Economics
Algorithms
Controls
Biology
Internet
Goal
NP
Communications
Dynamical Systems
Physics
• "handcrafted" theories are special cases
• immediate improvements
• new questions and answers

- The unifying language is (new) mathematics
- Tried to describe one important idea using simplest example and minimal math
- Just the beginning, but promising foundation for ASE?





Pedagogical strategy:

- Describe theorems on highly abstracted & simplified toy models that illustrate essence of general principles
- Nearly “math-free” exposition using cartoons and pictures

Physics